UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 12
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Tempered Distributions

Homework Problems

42. Solving PDEs via the Fourier transform

Assume $f \in L^1(\mathbb{R}^2)$ is such that also its Fourier transform $\hat{f}$ is integrable. Solve the PDE

$$\frac{\partial^2 u}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_2^2} + 3\frac{\partial u}{\partial x_1} - 4u = f$$

on $\mathbb{R}^2$ using the Fourier transform, and discuss the existence of the solution u.

Solution:

If u and all derivatives up to second order are integrable, we can alternatively solve the equation

$$\mathcal{F}\left(\frac{\partial^2 u}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_2^2} + 3\frac{\partial u}{\partial x_1} - 4u\right) = \big(-\xi_1^2 - 2\xi_2^2 + \mathrm{i}3\xi_1 - 4\big)\hat{u} = \hat{f}.$$

The polynomial $P(\xi) := -\xi_1^2 - 2\xi_2^2 + \mathrm{i}3\xi_1 - 4$ has no real zeros, and thus $1/P$ is bounded. Hence, we can solve for $\hat{u}$ and Fourier back-transform,

$$u(x) = \Big(\mathcal{F}^{-1}\big(\hat{f}/P\big)\Big)(x) = 2\pi\left(\mathcal{F}^{-1}(1/P)\right) * f(x).$$

The function u exists, because $\hat{f} \in L^1(\mathbb{R}^2)$ by assumption so that also $\hat{f}/P$ is integrable,

$$\big\|\hat{f}/P\big\|_1 = \int_{\mathbb{R}^2} \mathrm{d}\xi\, \frac{|\hat{f}(\xi)|}{|P(\xi)|} \leq \sup_{\xi\in\mathbb{R}^2} \big|P(\xi)\big|^{-1} \big\|\hat{f}\big\|_1 < \infty.$$

43. Computations involving distributions (38 points)

Consider the following tempered distributions $L \in \mathcal{S}'(\mathbb{R})$,

$$(L, \varphi) := \int_{\mathbb{R}} \mathrm{d}x \, L(x) \, \varphi(x) \qquad\qquad \forall \varphi \in \mathcal{S}(\mathbb{R}),$$

where

(i) $L = \delta(x)$ (ii) $L = x^2$ (iii) $L = \delta'_a(x) := \delta'(x-a), \; a \in \mathbb{R},$ (iv) $L = |x| \, .$

(a) Show that L is continuous.

(b) Compute the first two distributional derivatives of L.

(c) For (i)–(iii) compute the distributional Fourier transform of L.

Solution:

In what follows, φ is always an arbitrary $\mathcal{S}(\mathbb{R})$ function. To answer (a), we use the boundedness criterion from Proposition 7.2.2, a linear function $L : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathbb{C}$ is bounded if and only if

$$\bigl|(L, \varphi)\bigr| \leq C \sum_{|a|,|\alpha| \leq N} \|f\|_{a\alpha}$$

holds for some $C > 0$ and $N \in \mathbb{N}_0$. Alternatively, the right-hand side could be bounded by $C \max_{|a|,|\alpha| \leq N} \|f\|_{a\alpha}$

(i) (a) δ is clearly linear and since $|\delta(\varphi)| \leq \|\varphi\|_{00}$, the Dirac distribution is also continuous. [1]

(b) Let $\varphi \in \mathcal{S}(\mathbb{R})$. By definition first and second derivative are given by

$$\bigl(\delta', \varphi\bigr) = (-1) \; \bigl(\delta, \varphi'\bigr) = -\varphi'(0) \qquad [1]$$
$$\bigl(\delta'', \varphi\bigr) = (-1)^2 \; \bigl(\delta, \varphi''\bigr) = +\varphi''(0). \qquad [1]$$

(c) One can compute the Fourier transform explicitly:

$$\bigl(\mathcal{F}\delta, \varphi\bigr) \overset{[1]}{=} \bigl(\delta, \mathcal{F}\varphi\bigr) = (\mathcal{F}\varphi)(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x \, \varphi(x)$$
$$\overset{[1]}{=} \bigl((2\pi)^{-1/2}, \varphi\bigr)$$

Hence, the Fourier transform of δ is the constant function $(2\pi)^{-1/2}$.

(ii) (a) Since φ is a Schwartz function, $x^2 \varphi$ is integrable, and we have

$$\bigl|(x^2, \varphi)\bigr| \leq \int_{\mathbb{R}} \mathrm{d}x \, x^2 \, |\varphi(x)| = \bigl\|x^2 \, \varphi\bigr\|_{L^1(\mathbb{R})}, \qquad [1]$$

and thanks to Lemma 7.1.3 in the lecture notes, we can estimate the $L^1(\mathbb{R})$ norm by finitely many seminorms. This means x^2 defines a tempered distribution. [1]

(b) Using partial integration, we obtain the first distributional derivative:

$$\Bigl(\tfrac{\mathrm{d}}{\mathrm{d}x} x^2 \, , \, \varphi\Bigr) \overset{[1]}{=} -\bigl(x^2, \tfrac{\mathrm{d}}{\mathrm{d}x}\varphi\bigr) = -\int_{\mathbb{R}} \mathrm{d}x \, x^2 \, \varphi'(x)$$
$$\overset{[1]}{=} -\Bigl[x^2 \, \varphi(x)\Bigr]_{-\infty}^{+\infty} + \int_{\mathbb{R}} \mathrm{d}x \, \Bigl(\tfrac{\mathrm{d}}{\mathrm{d}x} x^2\Bigr) \, \varphi(x) = \int_{\mathbb{R}} \mathrm{d}x \, 2x \, \varphi(x) \overset{[1]}{=} \bigl(2x, \varphi\bigr).$$

The boundary terms vanish as φ decays to 0 faster than any polynomial. Hence, the distributional derivative coincides with the ordinary derivative.

The second derivative is computed analogously:

$$\begin{aligned}\Big(\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}x^2\,,\,\varphi\Big) &= \Big(\tfrac{\mathrm{d}}{\mathrm{d}x}2x\,,\,\varphi\Big) = -\big(2x,\varphi'\big) \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}x\,2x\,\varphi'(x) \\ &= -\Big[2x\,\varphi(x)\Big]_{-\infty}^{+\infty} + \int_{\mathbb{R}} \mathrm{d}x\,\Big(\tfrac{\mathrm{d}}{\mathrm{d}x}2x\Big)\,\varphi(x) \overset{[1]}{=} (2,\varphi).\end{aligned}$$

(c) The Fourier transform of x^2 in the sense of functions does not exist. Nevertheless, we can *compute* its distributional Fourier transform:

$$\begin{aligned}\big(\mathcal{F}x^2,\varphi\big) &\overset{[1]}{=} \big(x^2,\mathcal{F}\varphi\big) = \int_{\mathbb{R}} \mathrm{d}x\,x^2\,(\mathcal{F}\varphi)(x) \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}x\,\big(\mathcal{F}(-\mathrm{i})^2\partial_x^2\varphi\big)(x) \\ &= -\Big(1\,,\,\mathcal{F}\partial_x^2\varphi\Big) \overset{[1]}{=} \Big(-\mathcal{F}1\,,\partial_x^2\varphi\Big) \overset{[1]}{=} \Big(-\sqrt{2\pi}\,\delta,\partial_x^2\varphi\Big) \overset{[1]}{=} \Big(-\sqrt{2\pi}\,\delta'',\varphi\Big)\end{aligned}$$

(iii) (a) The tempered distribution δ_a' is defined as

$$(\delta_a',\varphi) = (-1)\,(\delta_a,\varphi') = -\int_{\mathbb{R}} \mathrm{d}x\,\delta(x-a)\,\varphi'(x) = -\varphi'(a). \qquad [1]$$

Hence, we can estimate $\big|\delta_a'(\varphi)\big| \leq \lVert\varphi\rVert_{01}$ and δ_a' is a tempered distribution. [1]

(b)

$$\begin{aligned}\big(\tfrac{\mathrm{d}}{\mathrm{d}x}\delta_a',\varphi\big) &= (-1)\;\big(\delta_a',\varphi'\big) = (-1)^2\;\big(\delta_a,\varphi''\big) = +\varphi''(a) &&[1]\\ \big(\tfrac{\mathrm{d}^2}{\mathrm{d}x^2}\delta_a',\varphi\big) &= (-1)^2\,\big(\delta_a',\varphi''\big) = (-1)^3\;\big(\delta_a,\varphi'''\big) = -\varphi'''(a) &&[1]\end{aligned}$$

(c)

$$\begin{aligned}\big(\mathcal{F}\delta_a',\varphi\big) &= \big(\delta_a',\mathcal{F}\varphi\big) \overset{[1]}{=} -\big(\delta_a,\tfrac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}\varphi\big) \overset{[1]}{=} +\big(\delta_a,\mathrm{i}\mathcal{F}(x\,\varphi)\big) = \mathrm{i}\big(\mathcal{F}(x\,\varphi)\big)(a) \\ &\overset{[1]}{=} \frac{\mathrm{i}}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{e}^{-\mathrm{i}ax}\,x\,\varphi(x) \overset{[1]}{=} \Big(\mathrm{i}(2\pi)^{-1/2}\,\mathrm{e}^{-\mathrm{i}ax}\,x\,,\,\varphi\Big)\end{aligned}$$

(iv) (a) Since the growth of $|x|$ is polynomially bounded, also $|x|\;\varphi$ is integrable and we obtain with the help of Lemma 7.1.3:

$$\begin{aligned}\big|\big(|x|\,,\varphi\big)\big| &\overset{[1]}{\leq} \int_{\mathbb{R}} \mathrm{d}x\,\big|x\,\varphi(x)\big| \overset{[1]}{=} \big\lVert x\,\varphi\big\rVert_{L^1(\mathbb{R})} \\ &\overset{[1]}{\leq} C_1\,\big\lVert x\,\varphi\big\rVert_{00} + C_2\,\max_{j\leq N}\big\lVert x\,\varphi\big\rVert_{j0} \\ &\overset{[1]}{\leq} C_1\,\lVert\varphi\rVert_{10} + C_2\,\max_{j\leq N}\lVert\varphi\rVert_{j+1,0}\end{aligned}$$

Hence, $|x|$ defines a tempered distribution.

(b) The function $|x|$ is differentiable everywhere but at the origin. We conjecture that the weak derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}x}\,|x| = \mathrm{sgn}(x) := \begin{cases} +1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

Let $\varphi \in \mathcal{S}(\mathbb{R})$. We write $\left(\frac{\mathrm{d}}{\mathrm{d}x}\left|x\right|, \varphi\right)$ as integral and split it up at $x = 0$,

$$\begin{aligned}\left(\tfrac{\mathrm{d}}{\mathrm{d}x}\left|x\right| \;,\; \varphi\right) &\overset{[1]}{=} -\left(\left|x\right|, \varphi'\right) = -\int_{\mathbb{R}} \mathrm{d}x\ \left|x\right|\ \varphi'(x) \\ &\overset{[1]}{=} -\int_{-\infty}^{0} \mathrm{d}x\,(-x)\,\varphi'(x) - \int_{0}^{+\infty} \mathrm{d}x\,(+x)\,\varphi'(x).\end{aligned}$$

After partial integration the right-hand side simplifies to

$$\begin{aligned}\ldots &\overset{[1]}{=} \int_{-\infty}^{0} \mathrm{d}x\,(-1)\,\varphi(x) + \int_{0}^{+\infty} \mathrm{d}x\,(+1)\,\varphi(x) \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}x\,\mathrm{sgn}(x)\,\varphi(x) \\ &\overset{[1]}{=} \left(\mathrm{sgn}, \varphi\right).\end{aligned}$$

Note that the boundary terms at $\pm\infty$ vanish.

We could have defined sgn arbitrarily at $x = 0$, because the integral does not change if we modify sgn on a set of measure zero.

The second derivative is computed analogously by splitting up the integral at 0 and using the Fundamental Theorem of Calculus:

$$\begin{aligned}\left(\tfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{sgn} \;,\; \varphi\right) &\overset{[1]}{=} -\left(\mathrm{sgn}, \varphi'\right) \\ &\overset{[1]}{=} -\int_{-\infty}^{0} \mathrm{d}x\,(-1)\cdot\varphi'(x) - \int_{0}^{+\infty} \mathrm{d}x\,(+1)\cdot\varphi'(x) \\ &= \Big[+\varphi(x)\Big]_{-\infty}^{0} - \Big[\varphi(x)\Big]_{0}^{+\infty} \\ &\overset{[1]}{=} 2\varphi(0) \overset{[1]}{=} \left(2\delta, \varphi\right)\end{aligned}$$

44. The Sokhotski-Plemelj formula

Consider the following linear maps on $\mathcal{S}(\mathbb{R})$:

$$\frac{1}{x \pm \mathrm{i}0}(\varphi) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}x \, \frac{\varphi(x)}{x \pm \mathrm{i}\varepsilon}$$

$$\big(\mathcal{P}\tfrac{1}{x}\big)(\varphi) := \lim_{\varepsilon \searrow 0} \int_{|x| \geq \varepsilon} \mathrm{d}x \, \frac{\varphi(x)}{x}$$

Derive the Sokhotski-Plemelj formula

$$\frac{1}{x \pm \mathrm{i}0} = \mathcal{P}\tfrac{1}{x} \mp \mathrm{i}\,\pi\,\delta.$$

Hint: Decompose the left-hand side into real and imaginary part.

Solution:

We split

$$\frac{1}{x \pm \mathrm{i}\varepsilon} = \frac{x}{x^2 + \varepsilon^2} \mp \mathrm{i}\, \frac{\varepsilon}{x^2 + \varepsilon^2}$$

into real and imaginary part. First, we discuss the contribution of the real part:

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}x \, \frac{x}{x^2 + \varepsilon^2}\, \varphi(x) &= \lim_{\varepsilon \searrow 0} \left(\int_0^{+\infty} \mathrm{d}x \, \frac{x}{x^2 + \varepsilon^2}\, \varphi(x) + \int_{-\infty}^{0} \mathrm{d}x \, \frac{x}{x^2 + \varepsilon^2}\, \varphi(x) \right) \\
&= \lim_{\varepsilon \searrow 0} \left(\int_0^{+\infty} \mathrm{d}x \, \frac{x}{x^2 + \varepsilon^2}\, \big(\varphi(x) - \varphi(-x)\big) \right)
\end{aligned}$$

In the second step, we have made a change of variables ($x \mapsto -x$). The integrand

$$f_\varepsilon(x) := \frac{x}{x^2 + \varepsilon^2}\, \big(\varphi(x) - \varphi(-x)\big)$$

is continuous and converges pointwise as $\varepsilon \searrow 0$ to the continuous, integrable function

$$f(x) = \frac{\varphi(x) - \varphi(-x)}{x}.$$

Continuity and integrability away from $x = 0$ are clear. Near the origin $x = 0$ we can use the mean value theorem to estimate f via φ', because

$$\begin{aligned}
f(x) = \frac{\varphi(x) - \varphi(-x)}{x} &= \frac{\varphi(x) - \varphi(0)}{x} - \frac{\varphi(-x) - \varphi(0)}{-x} \\
&\xrightarrow{\varepsilon \searrow 0} \varphi'(0) - \varphi'(0) = 0
\end{aligned}$$

reduces to the difference quotient of φ at $x = 0$.

Hence, $|f_\varepsilon(x)|$ is dominated by the integrable function $|f(x)|$ independently of $\varepsilon > 0$, and thus, we can use Dominated Convergence to interchange the limit $\varepsilon \searrow 0$ and integration:

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}x \, \frac{x}{x^2 + \varepsilon^2}\, \varphi(x) &= \int_0^{\infty} \mathrm{d}x \, \frac{\varphi(x) - \varphi(-x)}{x} \\
&= \lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{\infty} \mathrm{d}x \, \frac{\varphi(x) - \varphi(-x)}{x} \\
&= \lim_{\varepsilon \searrow 0} \left(\int_{\varepsilon}^{\infty} \mathrm{d}x \, \frac{\varphi(-x)}{-x} + \int_{\varepsilon}^{\infty} \mathrm{d}x \, \frac{\varphi(x)}{x} \right) \\
&= \lim_{\varepsilon \searrow 0} \left(\int_{-\infty}^{-\varepsilon} \mathrm{d}x \, \frac{\varphi(x)}{x} + \int_{+\varepsilon}^{+\infty} \mathrm{d}x \, \frac{\varphi(x)}{x} \right) \\
&= \big(\mathcal{P}\tfrac{1}{x}\big)(\varphi)
\end{aligned}$$

Now we discuss the contribution of the imaginary part:

$$\begin{aligned}
\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}x \, \frac{\varepsilon}{x^2 + \varepsilon^2} \, \varphi(x) &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}y \, \frac{\varepsilon^2}{\varepsilon^2 \, y^2 + \varepsilon^2} \, \varphi(\varepsilon y) \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}y \, \frac{1}{y^2 + 1} \, \varphi(\varepsilon y) \\
&= \varphi(0) \, \int_{\mathbb{R}} \mathrm{d}y \, \frac{1}{y^2 + 1}
\end{aligned}$$

In the last step, we have once again used Dominated Convergence ($\varphi \in \mathcal{S}(\mathbb{R})$). The last integral can be computed explicitly (or looked up in a table), and the value is π. Hence, we have shown

$$\frac{1}{x \pm \mathrm{i}0} = \mathcal{P}\tfrac{1}{x} \mp \mathrm{i} \, \pi \, \delta.$$