UNIVERSITY OF TORONTO

Foundations of Quantum Mechanics

(APM 421 H)

Winter 2014
Problem Sheet 12
(2014.11.28)

Homework for Bonus Points

This homework sheet is for bonus points. However, *all problems are immediately relevant for the final exam* and finishing them will be useful as part of the preparation independently of the bonus points. You need not finish all problems, anything you submit will count in your favor.

Homework Problems

37. The Banach space of trace class operators

Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded operator on an infinite-dimensional, separable Hilbert space. Then define the trace

$$\mathrm{Tr}\, A := \sum_{n=1}^{\infty} \langle \varphi_n, A\varphi_n \rangle$$

where $\{\varphi_n\}_{n\in\mathbb{N}}$ is an arbitrary orthonormal basis of $\mathcal{H}$. We have already shown that the definition of Tr does not depend on the choice of basis. The goal of this problem is to characterize elements of the Banach space of trace class operators

$$\mathcal{T}^1(\mathcal{H}) := \big\{ A \in \mathcal{B}(\mathcal{H}) \;\mid\; \mathrm{Tr}\, |A| < \infty \big\}$$

with norm $\|A\|_{\mathcal{T}^1} := \mathrm{Tr}\, |A|$. This will be done in steps:

(i) For an arbitrary $A \in \mathcal{B}(\mathcal{H})$, define $|A| := \sqrt{A^* A}$ via functional calculus. Prove that $|A| \geq 0$.

Now suppose $H \in \mathcal{B}(\mathcal{H})$ is bounded and selfadjoint so that $\sigma_{\mathrm{disc}}(H) = \{E_n\}_{n\in\mathcal{I}}$ and $\sigma_{\mathrm{ess}}(H) \subseteq \{0\}$. Here, the eigenvalues E_n are counted according to their multiplicities, and the index set $\mathcal{I}$ is $\mathbb{N}$ in case H has infinitely many eigenvalues and $\mathcal{I} = \{1, \ldots, N\}$ if the number of non-zero eigenvalues is finite.

(ii) Show that any eigenfunction of H and is an eigenfunction of $|H|$.

(iii) Prove $|\sigma(H)| = \sigma(|H|)$.

Now suppose $H \in \mathcal{B}(\mathcal{H})$ is just bounded and selfadjoint.

(iv) Use the Weyl Criterion to show that $\mathrm{Tr}\, |H| < \infty$ implies $\sigma_{\mathrm{ess}}(|H|) = \sigma_{\mathrm{ess}}(H) \subseteq \{0\}$.

(v) Show that for $H \in \mathcal{T}^1(\mathcal{H})$ we have

$$\|H\|_{\mathcal{T}^1} = \sum_{n=1}^{\infty} |E_n|$$

where $\sigma_{\mathrm{p}}(H) = \{E_n\}_{n\in\mathbb{N}}$ (i. e. E_n may be 0).

(vi) Let ρ be a density operator. Prove that $\sigma_{\mathrm{ess}}(\rho) \subseteq \{0\}$ and that $\sigma_{\mathrm{disc}}(\rho) \subseteq [0, 1]$.

38. Holomorphic functional calculus

Let $H = H^*$ be a *bounded* selfadjoint operator and f a function which is holomorphic in a neighborhood of $\sigma(H)$. Then we define $f(H)$ via *holomorphic functional calculus* via

$$f_\Gamma(H) := \frac{\mathrm{i}}{2\pi} \int_\Gamma \mathrm{d}z \, f(z) \, (H - z)^{-1}$$

where Γ is a contour which is contained in the region of holomorphy of f which encloses $\sigma(H)$.

(i) Prove that $f_\Gamma(H)$ coincides with $f(H)$ defined via functional calculus from Chapter 6.

(ii) Use functional calculus to prove that $f_\Gamma(H)$ does not depend on the choice of contour, i. e. if Γ' is another contour enclosing $\sigma(H)$, then $f_\Gamma(H) = f_{\Gamma'}(H)$.

(iii) Use one of the resolvent identities and results from complex analysis to prove

$$f_\Gamma(H) \, g_\Gamma(H) = (f \, g)_\Gamma(H).$$

Hint: Do *not* use functional calculus here (because then, the exercise is trivial).

39. Selfadjointness

Consider the Hamilton operator

$$H = -\Delta_x - \frac{1 - \cos |x|}{|x|^3}$$

endowed with domain $\mathcal{D}(H) = H^2(\mathbb{R}^3)$.

(i) Prove that H is selfadjoint.

(ii) How many eigenvalues does H have below 0? Justify your answer.

40. Hamilton operators for spin systems

Define the operator

$$H_0 = -\mathrm{i}\partial_{x_1} \otimes \sigma_1 - \mathrm{i}\partial_{x_2} \otimes \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i}\partial_{x_1} - \partial_{x_2} \\ -\mathrm{i}\partial_{x_1} + \partial_{x_2} & 0 \end{pmatrix}$$

equipped with domain $\mathcal{D}(H_0) = \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^2, \mathbb{C}^2)$ where σ_1 and σ_2 are the first two Pauli matrices.

(i) Show that H_0 is equivalent to a *hermitian*-matrix-valued multiplication operator.

(ii) Prove that H_0 is essentially selfadjoint.

(iii) Is the the selfadjoint extension $H := \overline{H_0}$ bounded from below? Justify your answer.

(iv) Show that $K := C\big(\mathrm{id}_{L^2(\mathbb{R}^2)} \otimes \sigma_2\big)$ commutes with H on $\mathcal{D}(H)$.

41. The discrete position operator

Let us define the position operator Q on $\ell^2(\mathbb{Z})$ via

$$(Q\psi)(n) := n \, \psi(n)$$

where the domain $\mathcal{D}(Q) \subseteq \ell^2(\mathbb{Z})$ has yet to be determined.

(i) Find the domain of selfadjointness for Q and verify that Q is then indeed selfadjoint.

(ii) Compute $\sigma(Q)$ and find all spectral decompositions $\sigma_\sharp(Q)$ where $\sharp$ stands for p, c, r, disc, ess, pp, sc and ac.

(iii) Compute the projection-valued measure $1_\Lambda(Q)$ explicitly where $\Lambda \subseteq \mathbb{R}$ is a Borel set.

Hand in home work on: Friday, 14 December 2014, before class