UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics (APM 351 Y)

2013–2014
Problem Sheet 12
(2014.01.16)

Tempered Distributions

Homework Problems

42. Solving PDEs via the Fourier transform

Assume $f \in L^1(\mathbb{R}^2)$ is such that also its Fourier transform $\hat{f}$ is integrable. Solve the PDE

$$\frac{\partial^2 u}{\partial x_1^2} + 2\frac{\partial^2 u}{\partial x_2^2} + 3\frac{\partial u}{\partial x_1} - 4u = f$$

on $\mathbb{R}^2$ using the Fourier transform, and discuss the existence of the solution u.

43. Computations involving distributions

Consider the following tempered distributions $L \in \mathcal{S}'(\mathbb{R})$,

$$(L, \varphi) := \int_{\mathbb{R}} \mathrm{d}x \, L(x) \, \varphi(x) \qquad \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}),$$

where

(i) $L = \delta(x)$ (ii) $L = x^2$ (iii) $L = \delta'_a(x) := \delta'(x-a), \; a \in \mathbb{R},$ (iv) $L = |x| \, .$

(a) Show that L is continuous.

(b) Compute the first two distributional derivatives of L.

(c) For (i)–(iii) compute the distributional Fourier transform of L.

44. The Sokhotski-Plemelj formula

Consider the following linear maps on $\mathcal{S}(\mathbb{R})$:

$$\frac{1}{x \pm \mathrm{i}0}(\varphi) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}x \, \frac{\varphi(x)}{x \pm \mathrm{i}\varepsilon}$$

$$\big(\mathcal{P}\tfrac{1}{x}\big)(\varphi) := \lim_{\varepsilon \searrow 0} \int_{|x| \geq \varepsilon} \mathrm{d}x \, \frac{\varphi(x)}{x}$$

Derive the Sokhotski-Plemelj formula

$$\frac{1}{x \pm \mathrm{i}0} = \mathcal{P}\tfrac{1}{x} \mp \mathrm{i} \, \pi \, \delta.$$

Hint: Decompose the left-hand side into real and imaginary part.

Hand in home work on: Thursday, 23 January 2014, before class