UNIVERSITY OF TORONTO

# Differential Equations of Mathematical Physics
**(APM 351 Y)**

2013–2014
Solutions 13
(2014.01.23)

## Review for Test 2

There is *no homework due next Thursday*. Test 2 covers everything from **Chapter 5.3 to the end of Chapter 7** as well as **exercise sheets 8–12**. The coordinates of Test 2 in space-time are

Thursday, 30 January, 9–11, in room SS 1074.

The *same rules* as last time apply: *no aids* of any kind are allowed. And please bring a *photo id*.
The most important topics are listed below, but please note that not all problems in Test 2 need to be included in the list.

### Fourier transforms

(1) You should be able to **compute** *discrete* and *continuous* Fourier transform of *functions and distributions* (cf. e. g. problems 30, 34, 39 and 43). In some cases, you need to make use of properties of the Fourier transform to compute Fourier transforms efficiently.

**Exercise:**

(i) Compute the discrete Fourier transform of 3 functions on $[-\pi, +\pi]$.

(ii) Compute the continuous Fourier transform of 3 functions on $\mathbb{R}$.

(iii) Compute the Fourier transform of 3 tempered distributions.

(2) You should know about the relation between decay of the Fourier transform and the regularity of the original function (see e. g. Chapters 6.1.3 and 6.2.2).

**Exercise:**

(i) Give decay estimates for the discrete Fourier transforms $f(x) = -\frac{x^2}{\pi}$ and $g(x) = \cos(\sqrt{3}\,x)$, seen as functions on $[-\pi, +\pi]$.

(ii) Find the largest $k$ so that $h \in \mathcal{C}^k([-\pi, +\pi])$ where $h(x) = \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{|n|^{7/2}} \, \mathrm{e}^{+\mathrm{i}nx}$.

(3) Review the $L^2$-*theory* of the Fourier transform, e. g. that $\mathcal{F} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is unitary and how the Fourier transform is defined on $L^2(\mathbb{R}^d)$.

**Exercise:** Prove that the discrete Fourier transform can be regarded as a unitary map

$$\mathcal{F} : L^2(\mathbb{T}^d) \longrightarrow \ell^2(\mathbb{Z}^d).$$

## Solutions of translation-invariant PDEs on $\mathbb{R}^d$, $[-\pi, +\pi]^d$

(1) You should know *how to solve the usual PDEs* (e. g. the homogeneous and inhomogeneous heat equations, the free Schrödinger equation and the wave equation) using the Fourier transform (see e. g. problems 33, 37, 40 and 42 as well as the Chapters 6.1.6 and 6.2.4, 6.2.5 and Chapter 7.3).

**Exercise:** Solve the inhomogeneous free Schrödinger equation

$$\mathrm{i}\,\partial_t \psi(t) = -\Delta\psi(t) + f(t).$$

(2) Analyze and understand the usual strategies to solve these PDEs (e. g. using the Fourier transform to convert the *P*DE into an *O*DE).

**Exercise:** Compile a list of all strategies for solving PDEs covered up until now.

(3) Review the tight-binding models from physics (Chapter 6.1.6.2 and problem 37).

## Computations involving tempered distributions

(1) Review how to extend operations such as taking derivatives, Fourier transforms and the convolution from $\mathcal{S}(\mathbb{R}^d)$ to the tempered distributions.

(2) *Review problem 42* and the examples in Chapter 7. *Make sure to be fluent in computations.*

(3) Understand how a function can be considered as a tempered distribution.

## Operators

(1) Review problems 27 and 28.

(2) Understand notions such as *multiplication operators* and the most common examples in physics (e. g. problem 27 and Chapter 6.1.6).

**Exercise:** Let $P = P^* = P^2 \neq 0$ be an orthogonal projection on a Hilbert space $\mathcal{H}$. Prove $P \geq 0$.

PS There will be no solutions.