UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 14
(2014.01.30)

Tempered Distributions & Green's Functions

Homework Problems

45. The Sokhotski-Plemelj formula

Consider the linear maps

$$\frac{1}{x \pm \mathrm{i}0}(\varphi) := \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \mathrm{d}x \, \frac{\varphi(x)}{x \pm \mathrm{i}\varepsilon}$$
$$\big(\mathcal{P}\tfrac{1}{x}\big)(\varphi) := \lim_{\varepsilon \searrow 0} \int_{|x| \geq \varepsilon} \mathrm{d}x \, \frac{\varphi(x)}{x}$$

on $\mathcal{S}(\mathbb{R})$ from problem 44. Show that they define tempered distributions.

Hint: Inspect the solution to problem 44 and work smart, not hard.

Solution:

It suffices to show that $\mathcal{P}\frac{1}{x}$ defines a tempered distribution, because then, $\frac{1}{x \pm \mathrm{i}0}$ as the sum of two distributions is automatically a distribution.

The linearity of $\mathcal{P}\frac{1}{x}$ is obvious, and we only need to show continuity: Inspecting the solution of exercise 44, we need to analyze

$$\big(\mathcal{P}\tfrac{1}{x}\big)(\varphi) = \int_0^{+\infty} \mathrm{d}x \, \frac{\varphi(x) - \varphi(-x)}{x}$$

more closely where $\varphi \in \mathcal{S}(\mathbb{R})$. We cut the integral into two pieces,

$$\int_0^{+\infty} \mathrm{d}x \, \frac{\varphi(x) - \varphi(-x)}{x} = \int_0^{1} \mathrm{d}x \, \frac{\varphi(x) - \varphi(-x)}{x} + \int_1^{+\infty} \mathrm{d}x \, \frac{\varphi(x) - \varphi(-x)}{x},$$

and start with the second term. Given that $x \geq 1$, we can estimate

$$\left| \frac{\varphi(x) - \varphi(-x)}{x} \right| \leq |\varphi(x)| + |\varphi(-x)|,$$

and thus, Lemma 7.1.3 yields a bound of

$$\left| \int_1^{+\infty} \mathrm{d}x \, \frac{\varphi(x) - \varphi(-x)}{x} \right| \leq \int_1^{+\infty} \mathrm{d}x \, \big(|\varphi(x)| + |\varphi(-x)| \big) \leq \int_{\mathbb{R}} \mathrm{d}x \, |\varphi(x)|$$
$$\leq C_1(1) \, \|\varphi\|_{00} + C_2(1) \max_{j=0,1,2} \|f\|_{j0}.$$

The first piece is estimated with the help of mean value theorem,

$$\left|\frac{\varphi(x)-\varphi(-x)}{0}\right| \leq \left|\frac{\varphi(x)-\varphi(0)}{x}\right| + \left|\frac{\varphi(-x)-\varphi(0)}{x}\right|$$
$$\leq \sup_{x\in[0,1]} |\varphi'(x)| + \sup_{x\in[-1,0]} |\varphi'(x)| \leq 2\,\|\varphi\|_{01},$$

because the integrand is bounded and the domain of integration compact,

$$\left|\int_0^1 \mathrm{d}x\,\frac{\varphi(x)-\varphi(-x)}{x}\right| \leq 2\,\|\varphi\|_{01}.$$

Overall, we have estimated

$$\left|\big(\mathcal{P}\tfrac{1}{x}\big)(\varphi)\right| \leq 2\,\|\varphi\|_{01} + C_1(1)\,\|\varphi\|_{00} + C_2(1)\,\max_{j=0,1,2}\|f\|_{j0}$$

by finitely many seminorms, and thus, $\mathcal{P}\frac{1}{x}$ is continuous by Proposition 7.2.2.

46. Green's function for the Poisson equation in $d = 2$ (13 points)

Modify the proof of Theorem 8.3.1 to the case $d = 2$ in order to prove that the Green's function for $-\Delta_x$ in two dimensions is $G(x, y) = -\frac{1}{2\pi} \ln |x - y|$.

Solution:

We will reuse the notation from the proof of Theorem 8.3.1. First of all, away from $x = 0$, let us compute $\Delta_x \ln |x|$. Clearly, $\ln |x|$ is symmetric under exchange of x_1 and x_2, and we only need to compute the first two partial derivatives with respect to x_j, $j = 1, 2$:

$$\partial_{x_j} \ln |x| = \frac{1}{|x|} \, \partial_{x_j} \left(\sqrt{x_1^2 + x_2^2} \right) \overset{[1]}{=} \frac{x_j}{|x|^2}$$

$$\partial_{x_j}^2 \ln |x| = \frac{1}{|x|^2} - 2 \, x_j^2 \left(x_1^2 + x_2^2 \right)^{-2} \overset{[1]}{=} \frac{x^2 - 2 \, x_j^2}{|x|^4}$$

Thus, adding the right-hand sides for $j = 1, 2$ yields

$$\Delta_x \ln |x| = \frac{x^2 - 2 \, x_1^2}{|x|^4} + \frac{x^2 - 2 \, x_2^2}{|x|^4} \overset{[1]}{=} 0.$$

Now let $\varphi \in \mathcal{S}(\mathbb{R}^2)$ be a test function. We apply the weak Laplacian to $\ln |x|$ and insert $0 = \Delta_x \ln |x| \; \varphi(x)$,

$$\Big(-\Delta_x \ln |x| \; , \; \varphi \Big) \overset{[1]}{=} -\Big(\ln |x| \; , \; \Delta_x \varphi \Big) \overset{[1]}{=} - \lim_{\varepsilon \searrow 0} \int_{V_\varepsilon} \mathrm{d}x \; \ln |x| \; \Delta_x \varphi(x)$$

$$\overset{[1]}{=} \lim_{\varepsilon \searrow 0} \int_{V_\varepsilon} \mathrm{d}x \, \Big(\Delta_x \ln |x| \; \varphi(x) - \ln |x| \; \Delta_x \varphi(x) \Big).$$

Again, Green's formula applies,

$$\ldots \overset{[1]}{=} \lim_{\varepsilon \searrow 0} \int_{\partial V_\varepsilon} \mathrm{d}S \cdot \Big(\partial_r \ln |x| \; \varphi(x) - \ln |x| \; \partial_r \varphi(x) \Big)$$

$$\overset{[1]}{=} - \lim_{\varepsilon \searrow 0} \int_{\partial B_\varepsilon} \mathrm{d}S \cdot \Big(\varepsilon^{-1} \, \varphi(x) - \ln \varepsilon \, \partial_r \varphi(x) \Big),$$

and we obtain an integral with respect to $\mathrm{d}S$, the line measure of the circle of radius ε [1]. Since the circumference of B_ε scales like ε, the second term vanishes ($\lim_{\varepsilon \to 0} \big(\varepsilon \, \ln \varepsilon \big) = 0$ [1]) while the first term converges to

$$\Big(-\Delta_x \ln |x| \; , \; \varphi \Big) \overset{[1]}{=} -\mathrm{Area}(\mathbb{S}^1) \, \varphi(0)$$

$$\overset{[1]}{=} -2\pi \, (\delta, \varphi).$$

Hence, the Green's function is given by $G(x, y) = -\frac{1}{2\pi} \ln |x - y|$ [1] and satisfies $-\Delta G(x, y) = \delta(x - y)$.