Quantum Mechanics

Homework Problems

47. Translations in real and momentum space

Let $T_a : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$, $(T_a\psi)(x) := \psi(x-a)$, be the translation operator by $a \in \mathbb{R}^d$ and $S_b : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ the translation operator in momentum space, defined for $b \in \mathbb{R}^d$ through

$$\big(\mathcal{F} S_b \psi\big)(\xi) := \big(\mathcal{F}\psi\big)(\xi - b).$$

(i) Prove that T_a and S_b are unitary and compute their adjoints.

(ii) Prove that S_b is the operator of multiplication by $\mathrm{e}^{+ib\cdot x}$.

(iii) Is $T_a S_b$ equal to $S_b T_a$?

Solution:

(i) Let $a \in \mathbb{R}^d$, $\varphi, \psi \in L^2(\mathbb{R}^d)$. To compute the adjoint operator, we plug everything into the scalar product:

$$\begin{aligned}
\big\langle \varphi, T_a\psi \big\rangle &= \int_{\mathbb{R}^d} \overline{\varphi(x)}\,(T_a\psi)(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \overline{\varphi(x)}\,\psi(x-a)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \overline{\varphi(y+a)}\,\psi(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} (\overline{T_{-a}\varphi})(y)\,\psi(y)\,\mathrm{d}y \\
&= \big\langle T_{-a}\varphi, \psi \big\rangle
\end{aligned}$$

Hence, we conclude $T_a^* = T_{-a}$. Obviously, $T_{-a} = T_a^{-1}$ is the inverse to T_a, because for all $\varphi \in L^2(\mathbb{R}^d)$ we have

$$\big(T_{-a}T_a\varphi\big)(x) = \big(T_a\varphi\big)\big(x-(-a)\big) = \varphi(x+a-a) = \varphi(x).$$

Analogously, one can show $T_a T_{-a} = \mathrm{id}_{L^2(\mathbb{R}^d)}$.

Now to translations S_b, $b \in \mathbb{R}^d$, in momentum space: by definition,

$$\big(\mathcal{F} S_b \varphi\big)(k) = \big(\mathcal{F}\varphi\big)(k-b) = \big(T_b \mathcal{F}\varphi\big)(k) \tag{1}$$

holds for all $\varphi \in L^2(\mathbb{R}^d)$. To compute S_b^*, we use Plancherel's theorem (P) twice: for $\varphi, \psi \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}
\big\langle \varphi, S_b\psi \big\rangle &\overset{\text{(P)}}{=} \big\langle \mathcal{F}\varphi, \mathcal{F}S_b\psi \big\rangle \overset{(1)}{=} \big\langle \mathcal{F}\varphi, T_b\mathcal{F}\psi \big\rangle = \big\langle T_{-b}\mathcal{F}\varphi, \mathcal{F}\psi \big\rangle \\
&\overset{(1)}{=} \big\langle \mathcal{F}S_{-b}\varphi, \mathcal{F}\psi \big\rangle \overset{\text{(P)}}{=} \big\langle S_{-b}\varphi, \psi \big\rangle.
\end{aligned}$$

In other words, we have shown $S_b^* = S_{-b}$. That $S_{-b} = S_b^{-1}$ is the inverse of S_b follows from the definition, equation (1) as well as $T_{-b} = T_b^{-1}$:

$$\mathcal{F} S_{-b} S_b \varphi \overset{(1)}{=} T_{-b} \mathcal{F} S_b \varphi \overset{(1)}{=} T_{-b} T_b \mathcal{F} \varphi = \mathcal{F} \varphi$$

Analogously, one shows $S_b S_{-b} = \mathrm{id}_{L^2(\mathbb{R}^d)}$.

(ii) Let $\varphi \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ be a Schwartz function and $b \in \mathbb{R}^d$. Then we can write the Fourier transform as an integral:

$$\begin{aligned}
\big(S_b \varphi\big)(x) &= \big(\mathcal{F}^{-1} \mathcal{F} S_b \varphi\big)(x) \overset{(1)}{=} \big(\mathcal{F}^{-1} T_b \mathcal{F} \varphi\big)(x) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} \mathrm{e}^{+\mathrm{i} x \cdot k} \, \big(T_b \mathcal{F} \varphi\big)(k) \, \mathrm{d}k = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} \mathrm{e}^{+\mathrm{i} x \cdot k} \, (\mathcal{F} \varphi)(k - b) \, \mathrm{d}k \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} \mathrm{e}^{+\mathrm{i} x \cdot (k' + b)} \, (\mathcal{F} \varphi)(k') \, \mathrm{d}k' = \mathrm{e}^{+\mathrm{i} b \cdot x} \, \big(\mathcal{F}^{-1} \mathcal{F} \varphi\big)(x) \\
&= \mathrm{e}^{+\mathrm{i} b \cdot x} \, \varphi(x)
\end{aligned}$$

Schwartz functions are dense in $L^2(\mathbb{R}^d)$ (Theorem 7.1.7), and hence this computation extends by density to all of $L^2(\mathbb{R}^d)$ (cf. Theorem 5.1.6).

(iii) Pick arbitrary $a \in \mathbb{R}^d$, $b \in \mathbb{R}^d$ and $\varphi \in L^2(\mathbb{R}^d)$. Then by definition of T_a and (ii), we have

$$\big(T_a S_b \varphi\big)(x) = (S_b \varphi)(x - a) = \mathrm{e}^{+\mathrm{i} b \cdot (x - a)} \, \varphi(x - a)$$

and

$$\big(S_b T_a \varphi\big)(x) = \mathrm{e}^{+\mathrm{i} b \cdot x} \, (T_a \varphi)(x) = e^{i b \cdot x} \, \varphi(x - a).$$

Hence, if $a \cdot b \neq 0$, the operators $T_a S_b$ and $S_b T_a$ differ by a phase. Quite generally, we have

$$T_a S_b = \mathrm{e}^{-\mathrm{i} a \cdot b} T_a S_b.$$

Remark: The reason why translations in space and momentum via a and b, $a \cdot b \neq 0$, do not commute lies with the non-commutativity if position and momentum operator along the same direction which generate translations in momentum and real space (the order is reversed).

48. The discrete Laplacian

Consider the Hilbert space of square-summable sequences on $\mathbb{Z}$,

$$\ell^2(\mathbb{Z}) := \Big\{ \psi : \mathbb{Z} \longrightarrow \mathbb{C} \;\big|\; \textstyle\sum_{n\in\mathbb{Z}} |\psi(n)|^2 < \infty \Big\},$$

endowed with scalar product

$$\langle \psi, \varphi \rangle := \sum_{n\in\mathbb{Z}} \overline{\psi(n)} \varphi(n).$$

For $a \in \mathbb{Z}$ let

$$T_a : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}),\; (T_a\psi)(n) := \psi(n-a)$$

be the translation operator and

$$\Delta : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}),\; (\Delta\psi)(n) := \psi(n+1) + \psi(n-1) - 2\psi(n)$$

the discrete Laplace operator.

(i) Compute T_a^* and prove that T_a is unitary.

(ii) Show that T_a and Δ commute, i. e. $[T_a, \Delta] := T_a\Delta - \Delta T_a = 0$.

(iii) Compute Δ^*.

(iv) Determine E_k so that

$$\psi_k(n) := \mathrm{e}^{+\mathrm{i}kn}, \qquad\qquad n \in \mathbb{Z}, k \in [-\pi, +\pi],$$

is an eigenvalue to the discrete Laplacian,

$$(\Delta\psi_k)(n) = E_k \psi_k(n).$$

Is ψ_k an element of $\ell^2(\mathbb{Z})$?

Solution:

(i) The proof that T_a is unitary is completely analogous to problem 47: let $\varphi, \psi \in \ell^2(\mathbb{Z})$ and $a \in \mathbb{Z}$. The adjoint operator T_a^* is then T_{-a},

$$\begin{aligned}\big\langle \varphi, T_a\psi \big\rangle &= \sum_{n\in\mathbb{Z}} \overline{\varphi(n)}\,(T_a\psi)(n) = \sum_{n\in\mathbb{Z}} \overline{\varphi(n)}\,\psi(n-a) = \sum_{k\in\mathbb{Z}} \overline{\varphi(k+a)}\,\psi(k) \\ &= \sum_{k\in\mathbb{Z}} \overline{(T_{-a}\varphi)(k)}\,\psi(k) = \big\langle T_{-a}\varphi, \psi \big\rangle.\end{aligned}$$

T_{-a} is also the inverse to T_a, since

$$\big(T_{-a}T_a\varphi\big)(n) = (T_a\varphi)(n+a) = \varphi(n+a-a) = \varphi(n)$$

holds for all $\varphi \in \ell^2(\mathbb{Z})$ and $n \in \mathbb{Z}$. This means T_a is unitary.

(ii) It suffices to show that the commutator vanishes pointwise:

$$\begin{aligned}\big(T_a\Delta\psi\big)(n) &= (\Delta\psi)(n-a) = \psi(n-a+1) + \psi(n-a-1) - 2\psi(n-a) \\ \big(\Delta T_a\psi\big)(n) &= (T_a\psi)(n+1) + (T_a\psi)(n-1) - 2(T_a\psi)(n) \\ &= \psi(n-a+1) + \psi(n-a-1) - 2\psi(n-a) = \big(T_a\Delta\psi\big)(n)\end{aligned}$$

Hence, $[T_a, \Delta]\psi = 0$ and T_a commutes with Δ.

(iii) We will see that the discrete Laplacian Δ is selfadjoint: for all $\varphi, \psi \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned}
\big\langle \varphi, \Delta\psi \big\rangle &= \sum_{n \in \mathbb{Z}} \overline{\varphi(n)} \, (\Delta\psi)(n) = \sum_{n \in \mathbb{Z}} \overline{\varphi(n)} \, \big(\psi(n+1) + \psi(n-1) - 2\psi(n)\big) \\
&= \sum_{n \in \mathbb{Z}} \overline{\varphi(n-1)} \, \psi(n) + \sum_{n \in \mathbb{Z}} \overline{\varphi(n+1)} \, \psi(n) - 2 \sum_{n \in \mathbb{Z}} \overline{\varphi(n)} \, \psi(n) \\
&= \sum_{n \in \mathbb{Z}} \overline{\big(\varphi(n-1) + \varphi(n+1) - 2\varphi(n)\big)} \, \psi(n) = \sum_{n \in \mathbb{Z}} \overline{(\Delta\varphi)(n)} \, \psi(n) \\
&= \big\langle \Delta\varphi, \psi \big\rangle ,
\end{aligned}$$

i. e. $\Delta^* = \Delta$.

(iv) We apply Δ to the sequence ψ_k with entries $\psi_k(n) = \mathrm{e}^{+\mathrm{i}kn}$, $k \in [-\pi, +\pi]$ and obtain

$$\begin{aligned}
(\Delta\psi_k)(n) &= \psi_k(n+1) + \psi_k(n-1) - 2\psi_k(n) = \mathrm{e}^{+\mathrm{i}k(n+1)} + \mathrm{e}^{+\mathrm{i}k(n-1)} - 2\mathrm{e}^{+\mathrm{i}kn} \\
&= \big(\mathrm{e}^{+\mathrm{i}k} + \mathrm{e}^{-\mathrm{i}k} - 2\big) \, \mathrm{e}^{+\mathrm{i}kn} = \big(2 \cos k - 2\big) \, \mathrm{e}^{+\mathrm{i}kn} =: E_k \, \psi_k(n).
\end{aligned}$$

Since $\lvert \psi_k(n) \rvert = \big\lvert \mathrm{e}^{\mathrm{i}kn} \big\rvert = 1$ is independent of $n \in \mathbb{Z}$, the sequence ψ_k cannot be square summable, because $\psi_k \in \ell^2(\mathbb{Z})$ necessarily implies $\lim_{\lvert n \rvert \to \infty} \psi_k(n) = 0$.

49. The scaling operator (16 points)

Define position and momentum operator in the adiabatic scaling

$$\mathsf{q} := \varepsilon \hat{x}, \qquad \mathsf{p} := -\mathrm{i}\nabla_x,$$

as well as position and momentum operator in ordinary scaling

$$\mathsf{Q} := \hat{x}, \qquad \mathsf{P} := -\mathrm{i}\varepsilon\nabla_x,$$

acting on $L^2(\mathbb{R}^d)$. Moreover, for $\varepsilon > 0$ and $\varphi \in L^2(\mathbb{R}^d)$ we define the *scaling operator*

$$\big(U_\varepsilon \varphi\big)(x) := \varepsilon^{d/2} \, \varphi(\varepsilon x).$$

(i) Show that a surjective map $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ between two Hilbert spaces which satisfies

$$\big\langle U\varphi, U\psi \big\rangle_{\mathcal{H}_2} = \big\langle \varphi, \psi \big\rangle_{\mathcal{H}_1}$$

for all $\varphi, \psi \in \mathcal{H}_1$ is unitary.

(ii) Show that $U_\varepsilon : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is unitary. Compute U_ε^*.

(iii) Show that q and p are unitary equivalent to Q and P, i. e.

$$U_\varepsilon \mathsf{Q} U_\varepsilon^{-1} = \mathsf{q}, \qquad U_\varepsilon \mathsf{P} U_\varepsilon^{-1} = \mathsf{p}.$$

Solution:

(i) **Remark:** Initially, the condition that U is surjective was missing in part (i).

Let $\varphi, \psi \in \mathcal{H}_1$. Then we deduce $U^* U = \mathrm{id}_{\mathcal{H}_1}$ from

$$\big\langle \varphi, \psi \big\rangle_{\mathcal{H}_1} \overset{[1]}{=} \big\langle U\varphi, U\psi \big\rangle_{\mathcal{H}_2} \overset{[1]}{=} \big\langle U^* U\varphi, \psi \big\rangle_{\mathcal{H}_1}.$$

Similarly, one obtains $UU^* = \mathrm{id}_{\mathcal{H}_2}$ from $\big\langle U^*\varphi, U^*\psi \big\rangle_{\mathcal{H}_1} = \big\langle \varphi, \psi \big\rangle_{\mathcal{H}_2}$ for $\varphi, \psi \in \mathcal{H}_2$ and the fact that U is surjective. [1] Thus, $U^* = U^{-1}$ and U is unitary.

(ii) Let $\varphi, \psi \in L^2(\mathbb{R}^d)$. Then a simple substitution of variables yields

$$\begin{aligned}\big\langle U_\varepsilon\varphi, U_\varepsilon\psi \big\rangle &\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}x \, \big(U_\varepsilon\varphi\big)^*(x) \, \big(U_\varepsilon\psi\big)(x) \overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}x \, \varepsilon^{d/2} \varphi^*(\varepsilon x) \, \varepsilon^{d/2} \psi(\varepsilon x) \\ &= \int_{\mathbb{R}^d} \mathrm{d}y \, \varphi^*(y) \, \psi(y) \overset{[1]}{=} \big\langle \varphi, \psi \big\rangle.\end{aligned}$$

By (i) the operator U_ε is unitary, so the adjoint is the inverse, $U_\varepsilon^* = U_\varepsilon^{-1}$ [1], and the inverse is given by

$$\big(U_\varepsilon^{-1}\varphi\big)(x) \overset{[1]}{=} \varepsilon^{-d/2} \, \varphi\big(x/\varepsilon\big).$$

(iii) Let $\varphi \in \mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Then also $U_\varepsilon \varphi \in \mathcal{S}(\mathbb{R}^d)$ is a Schwartz functions and we obtain

$$\begin{aligned}\big(U_\varepsilon \mathsf{Q} U_\varepsilon^{-1}\varphi\big)(x) &\overset{[1]}{=} \varepsilon^{d/2} \big(\mathsf{Q} U_\varepsilon^{-1}\varphi\big)(\varepsilon x) \overset{[1]}{=} \varepsilon^{d/2} \, \varepsilon x \, \big(U_\varepsilon^{-1}\varphi\big)(\varepsilon x) \\ &\overset{[1]}{=} \varepsilon x \, \varphi(x) \overset{[1]}{=} \big(\mathsf{q}\varphi\big)(x).\end{aligned}$$

Analogously, we obtain for the momentum operator

$$\begin{aligned}\big(U_\varepsilon \mathsf{P} U_\varepsilon^{-1}\varphi\big)(x) &\overset{[1]}{=} \varepsilon^{d/2} \big(\mathsf{P} U_\varepsilon^{-1}\varphi\big)(\varepsilon x) \overset{[1]}{=} \varepsilon^{d/2} \, (-i\varepsilon) \big(\nabla_x \big(U_\varepsilon^{-1}\varphi\big)\big)(\varepsilon x) \\ &\overset{[1]}{=} \varepsilon^{d/2} \, (-i\varepsilon) \tfrac{1}{\varepsilon} \nabla_x \big(\big(U_\varepsilon^{-1}\varphi\big)(\varepsilon x)\big) = (-i\nabla_x) \, \varphi(x) \overset{[1]}{=} \big(\mathsf{p}\varphi\big)(x).\end{aligned}$$