UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 15
(2014.02.06)

Quantum Mechanics

Homework Problems

47. Translations in real and momentum space

Let $T_a : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$, $(T_a\psi)(x) := \psi(x-a)$, be the translation operator by $a \in \mathbb{R}^d$ and $S_b : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ the translation operator in momentum space, defined for $b \in \mathbb{R}^d$ through

$$\big(\mathcal{F} S_b \psi\big)(\xi) := \big(\mathcal{F}\psi\big)(\xi - b).$$

(i) Prove that T_a and S_b are unitary and compute their adjoints.

(ii) Prove that S_b is the operator of multiplication by $\mathrm{e}^{+\mathrm{i} b\cdot x}$.

(iii) Is $T_a S_b$ equal to $S_b T_a$?

48. The discrete Laplacian

Consider the Hilbert space of square-summable sequences on $\mathbb{Z}$,

$$\ell^2(\mathbb{Z}) := \Big\{\psi : \mathbb{Z} \longrightarrow \mathbb{C} \;\big|\; \textstyle\sum_{n\in\mathbb{Z}} \lvert\psi(n)\rvert^2 < \infty\Big\},$$

endowed with scalar product

$$\langle \psi, \varphi\rangle := \sum_{n\in\mathbb{Z}} \overline{\psi(n)}\varphi(n).$$

For $a \in \mathbb{Z}$ let

$$T_a : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}),\; (T_a\psi)(n) := \psi(n-a)$$

be the translation operator and

$$\Delta : \ell^2(\mathbb{Z}) \longrightarrow \ell^2(\mathbb{Z}),\; (\Delta\psi)(n) := \psi(n+1) + \psi(n-1) - 2\psi(n)$$

the discrete Laplace operator.

(i) Compute T_a^* and prove that T_a is unitary.

(ii) Show that T_a and Δ commute, i. e. $[T_a, \Delta] := T_a\Delta - \Delta T_a = 0$.

(iii) Compute Δ^*.

(iv) Determine E_k so that

$$\psi_k(n) := \mathrm{e}^{+\mathrm{i}kn}, \qquad n \in \mathbb{Z}, k \in [-\pi, +\pi],$$

is an eigenvalue to the discrete Laplacian,

$$(\Delta\psi_k)(n) = E_k\psi_k(n).$$

Is ψ_k an element of $\ell^2(\mathbb{Z})$?

49. The scaling operator

Define position and momentum operator in the adiabatic scaling

$$\mathsf{q} := \varepsilon \hat{x}, \qquad \mathsf{p} := -\mathrm{i}\nabla_x,$$

as well as position and momentum operator in ordinary scaling

$$\mathsf{Q} := \hat{x}, \qquad \mathsf{P} := -\mathrm{i}\varepsilon\nabla_x,$$

acting on $L^2(\mathbb{R}^d)$. Moreover, for $\varepsilon > 0$ and $\varphi \in L^2(\mathbb{R}^d)$ we define the *scaling operator*

$$\big(U_\varepsilon \varphi\big)(x) := \varepsilon^{d/2}\, \varphi(\varepsilon x).$$

(i) Show that a map $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ between two Hilbert spaces which satisfies

$$\big\langle U\varphi, U\psi \big\rangle_{\mathcal{H}_2} = \big\langle \varphi, \psi \big\rangle_{\mathcal{H}_1}$$

for all $\varphi, \psi \in \mathcal{H}_1$ is unitary.

(ii) Show that $U_\varepsilon : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is unitary. Compute U_ε^*.

(iii) Show that q and p are unitary equivalent to Q and P, i. e.

$$U_\varepsilon \mathsf{Q} U_\varepsilon^{-1} = \mathsf{q}, \qquad U_\varepsilon \mathsf{P} U_\varepsilon^{-1} = \mathsf{p}.$$

Hand in home work on: Thursday, 13 February 2014, before class