UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 16
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Quantum Mechanics

Homework Problems

50. The discrete Laplacian

Let Δ be the discrete Laplacian from problem 48.

(i) Show that Δ is a bounded operator on $\ell^2(\mathbb{Z})$.

(ii) Compute the spectrum of Δ.

Hint: Revisit Chapter 6.1.6.

Solution:

(i) We recognize Δ as the one-dimensional version of the nearest-neighbor hopping hamiltonian given in equation (6.1.15), and thus

$$\Delta = \mathfrak{s} + \mathfrak{s}^* - 2.$$

After (discrete) Fourier transform, Δ becomes the operator of multiplication with the function

$$E(k) = 2 \cos k - 2.$$

Since E is bounded, $E(\hat{k})$ and thus also Δ is bounded by problem 24.

(ii) The spectrum of multiplication operators, in turn, is given by the range of the function, i. e. $\sigma(\Delta) = [-4, 0]$.

51. Rank-1 operators

Suppose $\varphi, \psi \neq 0$ are elements of a Hilbert space $\mathcal{H}$, and define the rank-1 operator $P = |\varphi\rangle\langle\psi|$ via

$$P\phi = \langle\psi, \phi\rangle \ \varphi.$$

(i) Find all eigenvectors and eigenvalues of P.

(ii) Compute $\sigma(P)$.

(iii) Determine the nature of the spectrum, i. e. determine $\sigma_{\text{ess}}(P)$, $\sigma_{\text{disc}}(P)$, $\sigma_{\text{cont}}(P)$ and $\sigma_{\text{p}}(P)$.

Solution:

(i) We can read off the eigenvalues from the form of the operator: the first eigenvector is φ with eigenvalue $\lambda := \langle\psi, \varphi\rangle$.

The other eigenvalue is 0, because for any vector ϕ perpendicular to ψ, we have $P\phi = 0$, and thus the eigenspace is

$$\ker P = \{\psi\}^{\perp}.$$

Now there are two cases: $\psi \perp \varphi$, and then also $\lambda = 0$ and the only eigenvalue is 0. Or $\langle\psi, \varphi\rangle \neq 0$ and P has two different eigenvalues.

(ii) Clearly, $\{0, \lambda\} \subseteq \sigma(P)$ where $\lambda = \langle\psi, \varphi\rangle$.

Since $\text{ran}\, P = \text{span}\{\varphi\}$ is a one-dimensional subspace, the operator $P - z$ is always invertible on $(\text{ran}\, P)^{\perp}$. On the one-dimensional subspace $\text{ran}\, P$, the operator is invertible if and only if $z \neq \lambda$. Hence, we have shown $\sigma(P) = \{0, \lambda\}$.

(iii) By the classification introduced in Chapter 9.3, we know that

$$\begin{aligned}
\sigma_{\text{cont}}(P) &= \emptyset \\
\sigma_{\text{p}}(P) &= \sigma(P) \\
\sigma_{\text{ess}}(P) &= \{0\} \\
\sigma_{\text{disc}}(P) &= \begin{cases} \{\lambda\} & \lambda \neq 0 \\ \emptyset & \lambda = 0 \end{cases}
\end{aligned}$$

because the eigenspace associated to $\lambda \neq 0$ is one-dimensional and $\ker P$ is infinite-dimensional.

52. Operator kernels (13 points)

Let $T \in \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ be a bounded operator on $L^2(\mathbb{R}^d)$. Then the operator kernel K_T is the tempered distribution which satisfies

$$(T\varphi)(x) = \int_{\mathbb{R}^d} \mathrm{d}y \, K_T(x, y) \, \varphi(y)$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

(i) Find the operator kernels for the following operators:

(a) $\mathrm{id}_{L^2(\mathbb{R}^d)}$

(b) $P = |\varphi\rangle\langle\psi|$ defined as problem 51 where $\varphi, \psi \in L^2(\mathbb{R}^d)$

(c) $\big(-\partial_x^2 + E\big)^{-1}$ where $E > 0$ and $d = 1$

(ii) Let $T, S \in \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$. Show that the operator kernel of TS satisfies

$$K_{TS}(x, z) = \int_{\mathbb{R}^d} \mathrm{d}y \, K_T(x, y) \, K_S(y, z).$$

(iii) Let $T \in \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ be an operator with operator kernel K_T. Find the operator kernel of T^*.

Solution:

(i) (a) $K_{\mathrm{id}_{L^2(\mathbb{R}^d)}}(x, y) = \delta(x - y)$ [1]

(b) $K_P(x, y) = \varphi(x) \, \overline{\psi(y)}$ [1]

(c) Here, we actually need to do a little work: the operator kernel $-\partial_x^2 + E$ is actually its *Green's function*, and thus the operator kernel is

$$G(x, y) \overset{[1]}{=} \sqrt{2\pi} \, \big(\mathcal{F}^{-1}(\xi^2 + E)^{-1}\big)(x - y) \overset{[2]}{=} \frac{\mathrm{e}^{-\sqrt{E}|x-y|}}{2\sqrt{E}}.$$

(ii)

$$\begin{aligned}
\big(TS\varphi\big)(x) &\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}y \, K_T(x, y) \, (S\varphi)(y) \\
&\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}y \int_{\mathbb{R}^d} \mathrm{d}z \, K_T(x, y) \, K_S(y, z) \, \varphi(z) \\
&\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}y \, \bigg(\int_{\mathbb{R}^d} \mathrm{d}z \, K_T(x, y) \, K_S(y, z) \bigg) \, \varphi(z) \\
&\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}y \, K_{TS}(x, z) \, \varphi(z)
\end{aligned}$$

(iii) From the definition of the adjoint, we get

$$\begin{aligned}
\big\langle T^*\varphi, \psi \big\rangle = \big\langle \varphi, T\psi \big\rangle &\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{\varphi(x)} \, (T\psi)(x) \\
&\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{\varphi(x)} \int_{\mathbb{R}^d} \mathrm{d}y \, K_T(x, y) \, \psi(y) \\
&\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}y \, \overline{\bigg(\int_{\mathbb{R}^d} \mathrm{d}x \, \overline{K_T(x, y)} \, \varphi(x) \bigg)} \, \psi(y).
\end{aligned}$$

Hence, we deduce $K_{T^*}(x, y) = \overline{K_T(y, x)}$ [1].

53. Projections

Consider the multiplication operator $P = p(\hat{x})$ on $L^2(\mathbb{R}^d)$ associated to the function

$$p(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

(i) Compute $\sigma(P)$.

(ii) Determine the nature of the spectrum, i. e. determine $\sigma_{\text{ess}}(P)$, $\sigma_{\text{disc}}(P)$, $\sigma_{\text{cont}}(P)$ and $\sigma_{\text{p}}(P)$.

(iii) Prove that P is an orthogonal projection.

Solution:

(i) First of all, P has two eigenvalues, namely 0 and 1: the eigenvectors to the eigenvalue 1 are functions which vanish almost everywhere on $(-\infty, 0)$. Similarly, eigenvectors to the eigenvalue 0 are functions which vanish almost everywhere on $[0, +\infty)$.

Since $(P - z)\varphi = 0$ means that $\big((P - z)\varphi\big)(x) = 0$ for almost all $x \in \mathbb{R}$. Thus, for all $z \neq 0, 1$ we have $(P - z)\varphi \neq 0$ for all $\varphi \neq 0$, i. e. $P - z$ is invertible as long as $z \neq 0, 1$, and we have shown $\sigma(P) = \{0, 1\}$.

(ii) The eigenspaces to both eigenvalues are infinite-dimensional, and thus

$$\begin{aligned} \sigma_{\text{p}}(P) &= \sigma(P) \\ \sigma_{\text{cont}}(P) &= \emptyset \\ \sigma_{\text{ess}}(P) &= \{0, 1\} \\ \sigma_{\text{disc}}(P) &= \emptyset \end{aligned}$$

(iii) p is a real-valued, bounded function, and hence, by problem 24 $P^* = P$. (Otherwise, one needs to show this by hand for this special case.)

To see $P^2 = P$, we note $p^2 = p$ in the sense of functions and conclude

$$(P^2\varphi)(x) = p(x)^2\, \varphi(x) = p(x)\, \varphi(x) = (P\varphi)(x).$$

Thus, $P = P^* = P^2$ is an orthogonal projection.