Quantum Mechanics

Homework Problems

50. The discrete Laplacian

Let Δ be the discrete Laplacian from problem 48.

(i) Show that Δ is a bounded operator on $\ell^2(\mathbb{Z})$.

(ii) Compute the spectrum of Δ.

Hint: Revisit Chapter 6.1.6.

51. Rank-1 operators

Suppose $\varphi, \psi \neq 0$ are elements of a Hilbert space $\mathcal{H}$, and define the rank-1 operator $P = |\varphi\rangle\langle\psi|$ via

$$P\phi = \langle \psi, \phi \rangle \; \varphi.$$

(i) Find all eigenvectors and eigenvalues of P.

(ii) Compute $\sigma(P)$.

(iii) Determine the nature of the spectrum, i. e. determine $\sigma_{\mathrm{ess}}(P)$, $\sigma_{\mathrm{disc}}(P)$, $\sigma_{\mathrm{cont}}(P)$ and $\sigma_{\mathrm{p}}(P)$.

52. Operator kernels

Let $T \in \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ be a bounded operator on $L^2(\mathbb{R}^d)$. Then the operator kernel K_T is the tempered distribution which satisfies

$$(T\varphi)(x) = \int_{\mathbb{R}^d} \mathrm{d}y \, K_T(x,y) \, \varphi(y)$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

(i) Find the operator kernels for the following operators:

(a) $\mathrm{id}_{L^2(\mathbb{R}^d)}$

(b) $P = |\varphi\rangle\langle\psi|$ defined as problem 51 where $\varphi, \psi \in L^2(\mathbb{R}^d)$

(c) $\big(-\partial_x^2 + E\big)^{-1}$ where $E > 0$ and $d = 1$

(ii) Let $T, S \in \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$. Show that the operator kernel of TS satisfies

$$K_{TS}(x,z) = \int_{\mathbb{R}^d} \mathrm{d}y \, K_T(x,y) \, K_S(y,z).$$

(iii) Let $T \in \mathcal{B}\big(L^2(\mathbb{R}^d)\big)$ be an operator with operator kernel K_T. Find the operator kernel of T^*.

53. Projections

Consider the multiplication operator $P = p(\hat{x})$ on $L^2(\mathbb{R}^d)$ associated to the function

$$p(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

(i) Compute $\sigma(P)$.

(ii) Determine the nature of the spectrum, i. e. determine $\sigma_{\mathrm{ess}}(P)$, $\sigma_{\mathrm{disc}}(P)$, $\sigma_{\mathrm{cont}}(P)$ and $\sigma_{\mathrm{p}}(P)$.

(iii) Prove that P is an orthogonal projection.

Hand in home work on: Thursday, 27 February 2014, before class