Quantum Mechanics

Homework Problems

59. The Landau hamiltonian

Consider $H^A = \big(-\mathrm{i}\nabla_x - A\big)^2$ in $d = 2$ where A is a vector potential to the constant magnetic field $B = \partial_{x_1} A_2 - \partial_{x_2} A_1 = \text{const.}$

(i) Show that the Landau vector potential

$$A_{\mathrm{L}}(x) = B \begin{pmatrix} -x_2 \\ 0 \end{pmatrix}$$

is a vector potential to $B = \text{const.}$

(ii) Show that A_{L} is gauge-equivalent to

$$A_{\mathrm{s}} = \frac{B}{2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

i. e. find a function ϕ so that $A_{\mathrm{s}} = A_{\mathrm{L}} + \nabla_x \phi$.

(iii) Prove $H^{A_{\mathrm{s}}} = \mathrm{e}^{+\mathrm{i}\phi} \, H^{A_{\mathrm{L}}} \, \mathrm{e}^{-\mathrm{i}\phi}$.

(iv) Show that the Landau hamiltonian is unitarily equivalent to a shifted harmonic oscillator

$$H_{\mathrm{osc}}(\hat{\xi}) := -\partial_x^2 + \big(B\hat{x} + \hat{\xi}\,\big)^2$$

acting on a dense subset of $L^2(\mathbb{R}^2)$.

Solution:

(i) $B(x) = \partial_{x_1} A_{\mathrm{L}\,2} - \partial_{x_2} A_{\mathrm{L}\,1} = -\partial_{x_2}\big(-B\,x_2\big) = B$

(ii) The difference of the two vector potentials is

$$A_{\mathrm{s}}(x) - A_{\mathrm{L}}(x) = \frac{B}{2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} - B \begin{pmatrix} -x_2 \\ 0 \end{pmatrix} = \frac{B}{2} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

and thus we can choose $\phi(x) = \frac{B}{2}\, x_1 x_2$.

(iii)

$$\begin{aligned}
\mathrm{e}^{+\mathrm{i}\phi} \, H^{A_{\mathrm{L}}} \, \mathrm{e}^{-\mathrm{i}\phi} \psi &= \mathrm{e}^{+\mathrm{i}\frac{B}{2}x_1 x_2} \Big(\big(-\mathrm{i}\partial_{x_1} + Bx_2\big)^2 - \partial_{x_2}^2 \Big) \big(\mathrm{e}^{-\mathrm{i}\frac{B}{2}x_1 x_2} \psi\big) \\
&= \mathrm{e}^{+\mathrm{i}\frac{B}{2}x_1 x_2} \, \mathrm{e}^{-\mathrm{i}\frac{B}{2}x_1 x_2} \Big(\big(-\mathrm{i}\partial_{x_1} + \mathrm{i}^2 \, \tfrac{B}{2} \, x_2 + Bx_2\big)^2 + \big(-\mathrm{i}\partial_{x_2} - \tfrac{B}{2} x_1\big)^2 \Big) \psi \\
&= H^{A_{\mathrm{s}}} \psi
\end{aligned}$$

(iv) Start with $H^{A_{\mathrm{L}}}$ and define the partial Fourier transform

$$\big(\mathcal{F}_1 \psi\big)(\xi, x_2) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}x_1 \, \mathrm{e}^{-\mathrm{i} \xi x_1} \, \psi(x_1, x_2)$$

which acts only on x_1. Then we can reduce

$$\mathcal{F}_1 \, H^{A_{\mathrm{L}}} \, \mathcal{F}^{-1} = -\partial_{x_2}^2 + \big(B x_2 + \hat{\xi}\,\big)^2 = H_{\mathrm{osc}}(\hat{\xi})$$

to the shifted harmonic oscillator hamiltonian.

60. Magnetic translations (17 points)

Consider the magnetic Schrödinger operator $H^A = (-\mathrm{i}\nabla_x - A)^2$ on $L^2(\mathbb{R}^3)$. Moreover, define *magnetic translations*

$$\big(T^A_y \psi\big)(x) := \mathrm{e}^{-\mathrm{i} \int_{[x,x+y]} A} \, \psi(x+y)$$

where $y \in \mathbb{R}^d$ and

$$\int_{[x,x+y]} A = \int_0^1 \mathrm{d}s \, y \cdot A(x+sy)$$

is the magnetic circulation along the line-segment $[x, x+y]$.

(i) Show the *kinetic momentum operator* $\mathsf{P}^A_j = -\mathrm{i}\partial_{x_j} - A_j$ commutes with magnetic translations along the x_j-direction.

(ii) Find the phase function $\mathrm{e}^{-\mathrm{i}\omega(x,y,z)}$ so that

$$\big(T^A_y T^A_z \psi\big)(x) = \mathrm{e}^{-\mathrm{i}\omega(x,y,z)} \, \big(T^A_{y+z}\psi\big)(x).$$

Give a physical interpretation of ω.

Hint: Use Stoke's Theorem.

(iii) Do magnetic translations commute?

Solution:

(i) We begin by computing

$$\begin{aligned}
\partial_{x_j} \int_{[x,x+y]} A &\overset{[1]}{=} \partial_{x_j} \int_0^1 \mathrm{d}s \, y_j \, A_j(x+sy) \overset{[1]}{=} \int_0^1 \mathrm{d}s \, y_j \, \partial_{x_j} A_j(x+sy) \\
&\overset{[1]}{=} \int_0^1 \mathrm{d}s \, \frac{\mathrm{d}}{\mathrm{d}s} A_j(x+sy) \overset{[1]}{=} A_j(x+y) - A_j(x).
\end{aligned}$$

This will give the crucial cancellation in the following calculation:

$$\begin{aligned}
\big(\mathsf{P}^A_j \, T^A_y \psi\big)(x) &\overset{[1]}{=} \Big(\big(-\mathrm{i}\partial_{x_j} - A_j(\hat{x})\big) \big(T^A_y \psi\big)\Big)(x) \\
&\overset{[1]}{=} \big(-\mathrm{i}\partial_{x_j} - A_j(x)\big) \Big(\mathrm{e}^{-\mathrm{i}\int_{[x,x+y]} A} \, \psi(x+y)\Big) \\
&\overset{[1]}{=} \mathrm{e}^{-\mathrm{i}\int_{[x,x+y]} A} \, \Big(-\mathrm{i}\partial_{x_j} \psi(x+y) - A_j(x) + \\
&\qquad + \mathrm{i}^2 \, \big(A_j(x+y) - A_j(x)\big) \, \psi(x+y)\Big) \\
&\overset{[1]}{=} \mathrm{e}^{-\mathrm{i}\int_{[x,x+y]} A} \, \Big(-\mathrm{i}\partial_{x_j} - A_j(x+y)\Big) \psi(x+y) \\
&\overset{[1]}{=} \Big(T^A_y \big(-\mathrm{i}\partial_{x_j} - A(\hat{x})\big)\psi\Big)(x) \overset{[1]}{=} \big(T^A_y \, \mathsf{P}^A_j \psi\big)(x)
\end{aligned}$$

In other words, we have shown $[T^A_y, \mathsf{P}_j] = 0$.

(ii) We obtain the phase factor $\mathrm{e}^{-\mathrm{i}\omega(x,y,z)}$ by executing the magnetic translations along y and z, and then adding and subtracting the missing phase factor so as to combine to a magnetic translation by $y + z$:

$$\begin{aligned}\big(T^A_y\, T^A_z \psi\big)(x) &\overset{[1]}{=} \mathrm{e}^{-\mathrm{i} \int_{[x,x+y]} A} \, \big(T^A_z \psi\big)(x+y) \\ &\overset{[1]}{=} \mathrm{e}^{-\mathrm{i} \int_{[x,x+y]} A} \, \mathrm{e}^{-\mathrm{i} \int_{[x+y,x+y+z]} A} \, \psi(x+y+z) \\ &\overset{[1]}{=} \mathrm{e}^{-\mathrm{i} \int_{[x,x+y]} A} \, \mathrm{e}^{-\mathrm{i} \int_{[x+y,x+y+z]} A} \, \mathrm{e}^{+\mathrm{i} \int_{[x,x+y+z]} A} \, \big(T^A_{y+z} \psi\big)(x) \\ &=: \mathrm{e}^{-\mathrm{i}\omega(x,y,z)} \, \big(T^A_{y+z} \psi\big)(x)\end{aligned}$$

The crucial point here is Stoke's theorem to convert the sum of the line integrals to an integral over the enclosed surface: concretely, the three line integrals trace the borders of the triangle $\Delta(x, x+y, x+y+z)$ with corners x, $x+y$ and $x+y+z$. Thus, the phase factor is the magnetic flux through the triangle,

$$\begin{aligned}\omega(x,y,z) &\overset{[1]}{=} \int_{[x,x+y]} A + \int_{[x+y,x+y+z]} A - \int_{[x,x+y+z]} A \\ &\overset{[1]}{=} \int_{\Delta(x,x+y,x+y+z)} \nabla_x \times A \overset{[1]}{=} \int_{\Delta(x,x+y,x+y+z)} B.\end{aligned}$$

(iii) No, because usually $\mathrm{e}^{-\mathrm{i}\omega(x,y,z)} \neq \mathrm{e}^{-\mathrm{i}\omega(x,z,y)}$, they are magnetic fluxes through triangles with different corners. [1]

61. Number of negative eigenvalues of a Schrödinger operator

Consider $H(V) := -\Delta_x + V$. Assume V and W are potentials so that (i) $H(V)$ and $H(W)$ define selfadjoint operators on a common domain $\mathcal{D}$ and (ii) $\sigma_{\text{ess}}\big(H(V)\big) = \sigma_{\text{ess}}\big(H(W)\big) = [0, +\infty)$. Define $E_n(V)$ for the operator $H(V)$ as in Chapter 9.3.3.2 and let $N(V)$ to be the number of negative eigenvalues of $H(V)$.

Show that $V \leq W$ implies $E_n(V) \leq E_n(W)$, $n \in \mathbb{N}_0$, as well as $N(V) \geq N(W)$.

Solution:

$V \leq W$ implies $H(V) \leq H(W)$. More precisely, for any $\psi \in \mathcal{D}$ we have

$$\big\langle \psi, H(V)\psi \big\rangle \leq \big\langle \psi, H(W)\psi \big\rangle,$$

and consequently

$$\begin{aligned} E_n(V) &= \sup_{\substack{\varphi_1,\ldots,\varphi_n \in \mathcal{D} \\ \langle \varphi_j, \varphi_k \rangle = \delta_{jk}}} \inf_{\substack{\psi \in \{\varphi_1,\ldots,\varphi_n\}^\perp \\ \|\psi\|=1}} \big\langle \psi, H(V)\psi \big\rangle \\ &\leq \sup_{\substack{\varphi_1,\ldots,\varphi_n \in \mathcal{D} \\ \langle \varphi_j, \varphi_k \rangle = \delta_{jk}}} \inf_{\substack{\psi \in \{\varphi_1,\ldots,\varphi_n\}^\perp \\ \|\psi\|=1}} \big\langle \psi, H(W)\psi \big\rangle = E_n(W). \end{aligned}$$

The number of eigenvalues is defined as

$$N(V) = \big\{ N \in \mathbb{N} \;\big|\; E_n(V) < 0 \; \forall n \leq N, \text{ and } E_n(V) = 0 \; \forall n > N \big\} \in \mathbb{N}_0 \cup \{+\infty\}.$$

Given that $E_n(V) = 0$ necessarily implies $E_n(W) = 0$ because the essential spectrum always starts at 0, and hence $N(V) \geq N(W)$.

62. Birman-Schwinger principle for potential without fixed sign

Consider the Schrödinger operator $H = -\Delta_x + V$ for a potential that does not have a fixed sign, i. e. $V \not\leq 0$. Moreover, define the signed square root

$$V^{1/2}(x) := \mathrm{sgn}\big(V(x)\big)\ \lvert V(x)\rvert^{1/2}$$

and for $E > 0$ the Birman-Schwinger operator

$$K_E := \lvert V\rvert^{1/2}\ \big(-\Delta_x + E\big)^{-1}\, V^{1/2}.$$

Show that H has an eigenvalue at $-E$ if and only if K_E has an eigenvalue at -1. (The difference in sign is deliberate in order to conform to established sign conventions.)

Solution:

Assume ψ is an eigenvector of H to $-E$. Then

$$H\psi = \big(-\Delta_x + V\big)\psi = -E\,\psi$$

is equivalent to

$$\big(-\Delta_x + E\big)\psi = -V\,\psi = -V^{1/2}\,\varphi$$

where we have defined $\varphi := \lvert V\rvert^{1/2}\,\psi$. As $E > 0$, $-E \not\in \sigma(-\Delta_x) = [0, +\infty)$ and so the operator on the left-hand side is invertible. Bringing it to the other side and multiplying both sides by $\lvert V\rvert^{1/2}$ yields

$$\lvert V\rvert^{1/2}\,\psi = \varphi = -\,\lvert V\rvert^{1/2}\ \big(-\Delta_x + E\big)^{-1}\, V^{1/2}\varphi.$$

Put another way, $-E$ is an eigenvalue of H with eigenfunction ψ if and only if -1 is an eigenvalue of K_E with eigenfunction $\varphi = \lvert V\rvert^{1/2}\,\psi$.