UNIVERSITY OF TORONTO

# Differential Equations of Mathematical Physics

**(APM 351 Y)**

2013–2014
Problem Sheet 18
(2014.03.13)

## Quantum Mechanics

**Homework Problems**

**59. The Landau hamiltonian**

Consider $H^A = \bigl(-\mathrm{i}\nabla_x - A\bigr)^2$ in $d = 2$ where $A$ is a vector potential to the constant magnetic field $B = \partial_{x_1} A_2 - \partial_{x_2} A_1 = \text{const.}$

(i) Show that the Landau vector potential

$$A_{\mathrm{L}}(x) = B \begin{pmatrix} -x_2 \\ 0 \end{pmatrix}$$

is a vector potential to $B = \text{const.}$

(ii) Show that $A_{\mathrm{L}}$ is gauge-equivalent to

$$A_{\mathrm{s}} = \frac{B}{2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

i. e. find a function $\phi$ so that $A_{\mathrm{s}} = A_{\mathrm{L}} + \nabla_x \phi$.

(iii) Prove $H^{A_{\mathrm{s}}} = \mathrm{e}^{+\mathrm{i}\phi} \, H^{A_{\mathrm{L}}} \, \mathrm{e}^{-\mathrm{i}\phi}$.

(iv) Show that the Landau hamiltonian is unitarily equivalent to a shifted harmonic oscillator

$$H_{\mathrm{osc}}(\hat{\xi}) := -\partial_x^2 + \bigl(B\hat{x} + \hat{\xi}\,\bigr)^2$$

acting on a dense subset of $L^2(\mathbb{R}^2)$.

**60. Magnetic translations**

Consider the magnetic Schrödinger operator $H^A = (-\mathrm{i}\nabla_x - A)^2$ on $L^2(\mathbb{R}^3)$. Moreover, define *magnetic translations*

$$\bigl(T_y^A \psi\bigr)(x) := \mathrm{e}^{-\mathrm{i}\int_{[x,x+y]} A} \, \psi(x+y)$$

where $y \in \mathbb{R}^d$ and

$$\int_{[x,x+y]} A = \int_0^1 \mathrm{d}s \, y \cdot A(x+sy)$$

is the magnetic circulation along the line-segment $[x, x+y]$.

(i) Show the *kinetic momentum operator* $\mathsf{P}_j^A = -\mathrm{i}\partial_{x_j} - A_j$ commutes with magnetic translations along the $x_j$-direction.

(ii) Find the phase function $\mathrm{e}^{-\mathrm{i}\omega(x,y,z)}$ so that

$$\bigl(T_y^A T_z^A \psi\bigr)(x) = \mathrm{e}^{-\mathrm{i}\omega(x,y,z)} \, \bigl(T_{y+z}^A \psi\bigr)(x).$$

Give a physical interpretation of $\omega$.

**Hint:** Use Stoke's Theorem.

(iii) Do magnetic translations commute?

**61. Number of negative eigenvalues of a Schrödinger operator**

Consider $H(V) := -\Delta_x + V$. Assume $V$ and $W$ are potentials so that (i) $H(V)$ and $H(W)$ define selfadjoint operators on a common domain $\mathcal{D}$ and (ii) $\sigma_{\mathrm{ess}}\big(H(V)\big) = \sigma_{\mathrm{ess}}\big(H(W)\big) = [0, +\infty)$. Define $E_n(V)$ for the operator $H(V)$ as in Chapter 9.3.3.2 and let $N(V)$ to be the number of negative eigenvalues of $H(V)$.

Show that $V \leq W$ implies $E_n(V) \leq E_n(W)$, $n \in \mathbb{N}_0$, as well as $N(V) \geq N(W)$.

**62. Birman-Schwinger principle for potential without fixed sign**

Consider the Schrödinger operator $H = -\Delta_x + V$ for a potential that does not have a fixed sign, i. e. $V \not\leq 0$. Moreover, define the signed square root

$$V^{1/2}(x) := \mathrm{sgn}\big(V(x)\big)\ \lvert V(x)\rvert^{1/2}$$

and for $E > 0$ the Birman-Schwinger operator

$$K_E := \lvert V\rvert^{1/2}\ \big(-\Delta_x + E\big)^{-1}\, V^{1/2}.$$

Show that $H$ has an eigenvalue at $-E$ if and only if $K_E$ has an eigenvalue at $-1$. (The difference in sign is deliberate in order to conform to established sign conventions.)

**Hand in home work on:** Thursday, 13 March 2014, before class