UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 19
(2014.01.23)

Review for Test 3

There is *no homework due next Thursday*. The coordinates of Test 3 in space-time are

Thursday, 30 March, 9–11, in room SS 1074.

The *same rules* as last time apply: *no aids* of any kind are allowed. And please bring a *photo id*.

Test 3 covers everything from **Chapter 8 to the end of Chapter 9** as well as **exercise sheets 14–18**. However, in certain places it relies crucially on material from earlier chapters, in particular Chapters 4 (Hilbert and Banach spaces), 5 (linear operators), 6 (Fourier transforms) and 7 (tempered distributions). The most important topics are listed below, but please note that not all problems in Test 3 need to be included in that list.

Tempered distributions

The chapter on Green's functions builds on top of Chapter 7 (tempered distributions). *You should be fluent in computations with tempered distributions* including weak derivatives and weak Fourier transforms. This is crucial, because the fundamental equation $LG(x,y) = \delta(x-y)$ involves distributions.

Exercise: Review problem 43.

Green's functions

(1) You should be able to *compute* Green's functions to differential operators (see e. g. problem 42). You usually need not evaluate the Fourier transform.

(2) Study the Poisson equation on $\mathbb{R}^d$ and a bounded domain on $\mathbb{R}^2$.

Exercise:

(i) Repeat problem 42.

(ii) Show that $G(x,y) = C_d \, |x-y|^{d-2}$ is the Green's function for $L = \Delta_x$ in $d \neq 2$ and C_d is a constant that depends on the dimension. Make sure you do the computation on the level of distributions.

Operators

Review Chapters 4 and 5 and the associated homework assignments. For instance, exercises 27 and 28 on sheet 8 foreshadow Chapter 9. Notions such as selfadjoint, symmetric and unitary operators are crucial for understanding quantum mechanics. You should know what the spectrum is (cf. Chapters 5.1 and 9.3) and its decompositions into spectral components.

Exercises:

(i) Let $P \neq 0$ be an orthogonal projection on a Hilbert space $\mathcal{H}$. Prove $P \geq 0$.

(ii) Show that $H = -\Delta_x + V$ is a symmetric operator on $\mathcal{C}^\infty_{\text{per}}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$.

(iii) Review problems 24, 27 and 28.

Quantum mechanics

(1) Revisit Chapters 9.1–9.2. Study up on the fundamentals of quantum mechanics (states, obersvables and dynamical equations).

Exercise:

(i) What physical consequences does the selfadjointness of the Hamilton operator have?

(ii) What is the physical significance of the unitarity of $\mathrm{e}^{-\mathrm{i}tH}$?

(iii) Give states, observables and dynamical equation for the spin-$1/2$ system from Chapter 9.1.

(2) Revisit Heisenberg's Uncertainty Relation (Chapter 6.2.5 and problem 47).

Exercise: Review problem 54 and explain its connection to Heisenberg's uncertainty relation.

(3) Understand notions such as *multiplication operators* and the most common examples in physics (e. g. problem 27 and Chapter 6.1.6).

(4) Review Chapter 9.3. You should know how to apply the Birman-Schwinger and the min-max principle, for instance. You should know what bound states are and where the corresponding energy eigenvalues typically appear in the spectrum. I do *not* expect you to be able to make a proof akin to Theorem 9.3.7.

(5) You should be able to compute spectra of simple operators (see e. g. problems 50, 51, 53 and 55).

Exercise:

Consider the multiplication operator V defined by $V(x) := \max\{0, x\}$. Compute the spectrum, determine the nature of the spectra and find all eigenvectors if they exist (see e. g. problems 24, 51 and 53). Is V symmetric?

PS There will be no solutions.