UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 20
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Functionals

Homework Problems

63. The Abraham model (21 points)

The Abraham model describes a particle with a rigid charged density $\chi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$, $\int_{\mathbb{R}^3} \mathrm{d}x \, \chi(x) = 1$, coupled to an electromagnetic field. Here, rigid means that the charge distribution does not change in time. Moreover, we define $\phi_\chi = \chi * \phi$ and $A_\chi = \chi * A$ to be the smoothened potentials obtained by convolving ϕ and the components of A with the charge density χ.

The Lagrange function of this system is

$$L\big(q(t), \dot{q}(t), A(t), \phi(t), \dot{A}(t), \dot{\phi}(t)\big) := L_p\big(q(t), \dot{q}(t), A(t), \phi(t)\big) + L_{\mathrm{em}}\big(A(t), \phi(t), \dot{A}(t), \dot{\phi}(t)\big)$$

which is comprised of the particle Lagrangian

$$L_p(x, v, A, \phi) = \frac{m}{2} v^2 - \phi_\chi(x) + v \cdot A_\chi(x)$$

and the field Lagrangian

$$L_{\mathrm{em}}(A, \phi, \dot{A}, \dot{\phi}) = \frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}x \, \Big(\big(-\partial_t A(t,x) - \nabla_x \phi(t,x)\big)^2 - \big(\nabla_x \times A(t,x)\big)^2 \Big).$$

(i) Verify that $\rho(t,x) = \chi\big(x - q(t)\big)$ and $j(t,x) = \chi\big(x - q(t)\big) \, \dot{q}(t)$ satisfy charge conservation.

(ii) Give the action functional associated to the Lagrange function L.

(iii) Derive the Euler-Lagrange equations.

(iv) Compute the equations of motion in terms of the electric field **E** and magnetic field **B**.

Hint: The equations of motion also involve $\mathbf{E}_\chi = \chi * \mathbf{E}$ and $\mathbf{B}_\chi = \chi * \mathbf{B}$.

Note: Coupling the Maxwell and the Newton equations for a particle with point charge $\chi = \delta$ leads to equations which are ill-defined, and it turns out to be necessary to smear out the charge over a small region.

The interested reader may read the discussion in Chapter 2–2.4 of Herbert Spohn's book "Dynamics of Charged Particles", Cambridge University Press, 2004.

Solution:

Unfortunately there was a sign mistake ($-A \cdot v$ rather than $+A \cdot v$ in L_p).

(i) We compute the time derivative of the charge density,

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t,x) = \frac{\mathrm{d}}{\mathrm{d}t}\chi\big(x-q(t)\big) \overset{[1]}{=} -\nabla_x\chi\big(x-q(t)\big)\,\dot{q}(t),$$

and the divergence of the current density,

$$\nabla_x \cdot j(t,x) \overset{[1]}{=} \nabla_x\chi\big(x-q(t)\big)\,\dot{q}(t),$$

and see that they are equal up to a sign, $\partial_t\rho + \nabla_x \cdot j = 0$ [1].

(ii) $S(q,A,\phi) = \int_0^T \mathrm{d}t\,\Big(L_p\big(q(t),\dot{q}(t),A(t),\phi(t)\big) + L_{\mathrm{em}}\big(A(t),\phi(t),\dot{A}(t),\dot{\phi}(t)\big)\Big)$ [2]

(iii) Let us compute the Euler-Lagrange equations from first principles:

$$\begin{aligned}
&\big(\mathrm{d}S(q,A,\phi)\big)(h,a,\varphi) = \\
&\overset{[1]}{=} \int_0^T \mathrm{d}t\,\frac{\mathrm{d}}{\mathrm{d}s}\bigg(L_p\Big(q(t)+s\,h(t),\dot{q}(t)+s\,\dot{h}(t),A(t)+s\,a(t),\phi(t)+s\,\varphi(t)\Big)+ \\
&\qquad +L_{\mathrm{em}}\Big(A(t)+s\,a(t),\phi(t)+s\,\varphi(t),\dot{A}(t)+s\,\dot{a}(t),\dot{\phi}(t)+s\,\dot{\varphi}(t)\Big)\bigg)\bigg|_{s=0} \\
&\overset{[3]}{=} \int_0^T \mathrm{d}t\,\bigg(\Big(m\,\dot{q}(t)+A_\chi\big(t,q(t)\big)\Big)\cdot\dot{h}(t) - \nabla_x\phi_\chi\big(t,q(t)\big)\cdot h(t)+ \\
&\qquad +\sum_{j=1}^{3}\dot{q}_j(t)\,\nabla_x a_{\chi,j}\big(t,q(t)\big)\cdot h(t) - \varphi_\chi\big(t,q(t)\big) + \dot{q}(t)\cdot a_\chi\big(t,q(t)\big)\bigg)+ \\
&\quad +\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}x\,\Big(\big(-\partial_t A(t,x)-\nabla_x\phi(t,x)\big)\cdot\big(-\partial_t a(t,x)-\nabla_x\varphi(t,x)\big)+ \\
&\qquad -\big(\nabla_x\times A(t,x)\big)\cdot\big(\nabla_x\times a(t,x)\big)\Big) \\
&\overset{*,[3]}{=} -\sum_{k=1}^{3}\int_0^T \mathrm{d}t\,\bigg(m\,\ddot{q}_k(t)+\partial_t A_{\chi,k}\big(t,q(t)\big)+\partial_{x_k}\phi_\chi\big(t,q(t)\big)+ \\
&\qquad +\sum_{j=1}^{3}\big(\partial_{x_k}a_{\chi,j}-\partial_{x_j}a_{\chi,k}\big)\big(t,q(t)\big)\,\dot{q}_j(t)\bigg)\cdot h_k(t)+ \\
&\quad +\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}x\,\Big(-\varphi(t,x)\,\chi\big(x-q(t)\big)+\chi\big(x-q(t)\big)\,\dot{q}(t)\cdot a(t,x)\Big)\cdot h(t)+ \\
&\quad +\int_0^T \mathrm{d}t\int_{\mathbb{R}^3}\mathrm{d}x\,\Big(\big(-\partial_t^2 A(t,x)-\nabla_x\partial_t\phi(t,x)-\nabla_x\times\nabla_x\times A(t,x)\big)\cdot a(t,x)+ \\
&\qquad +\nabla_x\cdot\big(-\partial_t A(t,x)-\nabla_x\phi(t,x)\big)\,\varphi(t,x)\Big)
\end{aligned}$$

In the step marked with $*$ we have plugged in the definition of $a_\chi = \chi * a$ and $\varphi_\chi = \chi * \varphi$ as a convolution with the charge cloud χ. By plugging in the definition of ρ, j, **E** and **B**, we obtain a more concise expression:

$$
\begin{aligned}
\ldots \overset{[3]}{=} -\int_0^T \mathrm{d}t \left(m\,\ddot{q}(t) - \mathbf{E}_\chi\big(t,q(t)\big) - \dot{q}(t) \times \mathbf{B}_\chi\big(t,q(t)\big) \right) \cdot h(t) + \\
+ \int_0^T \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}x \, \Big(\big(\partial_t E(t,x) - \nabla_x \times \mathbf{B}(t,x) + j(t,x)\big) \cdot a(t,x) + \\
+ \big(\nabla_x \cdot \mathbf{E}(t,x) - \rho(t,x)\big)\, \varphi(t,x) \Big) \\
\overset{!}{=} 0
\end{aligned}
$$

That means we obtain the following coupled Euler-Lagrange equations:

$$
\begin{aligned}
m\,\ddot{q}(t) &\overset{[1]}{=} \mathbf{E}_\chi\big(t,q(t)\big) + \dot{q}(t) \times \mathbf{B}_\chi\big(t,q(t)\big) \\
\partial_t \mathbf{E} &\overset{[1]}{=} \nabla_x \times \mathbf{B} - j \\
\nabla_x \cdot \mathbf{E} &\overset{[1]}{=} \rho
\end{aligned}
$$

(iv) Clearly, the magnetic field $\nabla_x \cdot \mathbf{B} = \nabla_x \cdot \big(\nabla_x \times A\big) = 0$ [1] is divergence-free by definition. The equation of motion for **B** can be derived just like in Chapter 10.1.4.2:

$$
\begin{aligned}
\partial_t \mathbf{B} &\overset{[1]}{=} \nabla_x \times \partial_t A = -\nabla_x \times \big(-\partial_t A - \nabla_x \phi\big) \\
&\overset{[1]}{=} -\nabla_x \times \mathbf{E}
\end{aligned}
$$

64. Extrema under constraints

Consider the Schrödinger operator $H = -\Delta_x + V$ on $\mathbb{R}^d$ and assume $\sigma_{\mathrm{p}}(H) \neq \emptyset$.
Find the extremal points of the energy functional

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^d} \mathrm{d}x \, \Big(\big|\nabla_x \psi(x)\big|^2 + V(x) \, |\psi(x)|^2 \Big)$$

under the constraint

$$\mathcal{J}(\psi) = \int_{\mathbb{R}^d} \mathrm{d}x \, |\psi(x)|^2 - 1 = 0.$$

Solution:

We start by computing the Gâteaux derivative of the energy functional,

$$\begin{aligned}
\big(\mathrm{d}\mathcal{E}(\psi)\big)\varphi &= \int_{\mathbb{R}^d} \mathrm{d}x \, \frac{\mathrm{d}}{\mathrm{d}s} \Big(\big|\nabla_x \psi(x) + s \, \nabla_x \varphi(x)\big|^2 + V(x) \, |\psi(x) + s\varphi(x)|^2 \Big) \Big|_{s=0} \\
&= \int_{\mathbb{R}^d} \mathrm{d}x \, \Big(\big(\overline{\nabla_x \varphi(x)} \cdot \nabla_x \psi(x) + \overline{\nabla_x \psi(x)} \cdot \nabla_x \varphi(x) \big) + V(x) \, \big(\overline{\varphi(x)} \, \psi(x) + \overline{\psi(x)} \, \varphi(x) \big) \Big) \\
&= 2\mathrm{Re} \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{\varphi(x)} \, \big(-\Delta_x \psi(x) + V(x) \, \psi(x) \big),
\end{aligned}$$

and the derivative of the constraint,

$$\begin{aligned}
\big(\mathrm{d}\mathcal{J}(\psi)\big)\varphi &= \frac{\mathrm{d}}{\mathrm{d}s} \int_{\mathbb{R}^d} \mathrm{d}x \, \big|\psi(x) + s\varphi(x)\big|^2 - 1 \Big|_{s=0} \\
&= \int_{\mathbb{R}^d} \mathrm{d}x \, \big(\overline{\varphi(x)} \, \psi(x) + \overline{\psi(x)} \, \varphi(x) \big) \\
&= 2\mathrm{Re} \int_{\mathbb{R}^d} \mathrm{d}x \, \overline{\varphi(x)} \, \psi(x).
\end{aligned}$$

Applying the method of Lagrange multipliers, we obtain the equation

$$\begin{aligned}
\big(\mathrm{d}\mathcal{E}(\psi)\big)\varphi + \lambda \, \big(\mathrm{d}\mathcal{J}(\psi)\big)\varphi &= 2\mathrm{Re} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\varphi(x)} \, \big(-\Delta_x \psi(x) + V(x) \, \psi(x) + \lambda \psi(x) \big) \\
&= 2\mathrm{Re} \int_{\mathbb{R}^3} \mathrm{d}x \, \overline{\varphi(x)} \, \big(-\Delta_x + V(x) + \lambda \big) \psi(x) \overset{!}{=} 0.
\end{aligned}$$

That means

$$H\psi = -\lambda \, \psi,$$

and thus, any eigenvector ψ to E_ψ satisfies the equation for $\lambda = -E_\psi$.
The minimizer is obviously the ground state.