UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 20
(2014.03.20)

Functionals

Homework Problems

63. The Abraham model

The Abraham model describes a particle with a rigid charged density $\chi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$, $\int_{\mathbb{R}^3} \mathrm{d}x \, \chi(x) = 1$, coupled to an electromagnetic field. Here, rigid means that the charge distribution does not change in time. Moreover, we define $\phi_\chi = \chi * \phi$ and $A_\chi = \chi * A$ to be the smoothened potentials obtained by convolving ϕ and the components of A with the charge density χ.

The Lagrange function of this system is

$$L\big(q(t), \dot{q}(t), A(t), \phi(t), \dot{A}(t), \dot{\phi}(t)\big) := L_p\big(q(t), \dot{q}(t), A(t), \phi(t)\big) + L_{\mathrm{em}}\big(A(t), \phi(t), \dot{A}(t), \dot{\phi}(t)\big)$$

which is comprised of the particle Lagrangian

$$L_p\big(x, v, A, \phi\big) = \frac{m}{2} v^2 - \phi_\chi(x) + v \cdot A_\chi(x)$$

and the field Lagrangian

$$L_{\mathrm{em}}\big(A, \phi, \dot{A}, \dot{\phi}\big) = \frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}x \, \Big(\big(-\partial_t A(t,x) - \nabla_x \phi(t,x)\big)^2 - \big(\nabla_x \times A(t,x)\big)^2 \Big).$$

(i) Verify that $\rho(t,x) = \chi\big(x - q(t)\big)$ and $j(t,x) = \chi\big(x - q(t)\big) \, \dot{q}(t)$ satisfy charge conservation.

(ii) Give the action functional associated to the Lagrange function L.

(iii) Derive the Euler-Lagrange equations.

(iv) Compute the equations of motion in terms of the electric field **E** and magnetic field **B**.

Hint: The equations of motion also involve $\mathbf{E}_\chi = \chi * \mathbf{E}$ and $\mathbf{B}_\chi = \chi * \mathbf{B}$.

Note: Coupling the Maxwell and the Newton equations for a particle with point charge $\chi = \delta$ leads to equations which are ill-defined, and it turns out to be necessary to smear out the charge over a small region.

The interested reader may read the discussion in Chapter 2–2.4 of Herbert Spohn's book "Dynamics of Charged Particles", Cambridge University Press, 2004.

64. Extrema under constraints

Consider the Schrödinger operator $H = -\Delta_x + V$ on $\mathbb{R}^d$ and assume $\sigma_{\mathrm{p}}(H) \neq \emptyset$.

Find the extremal points of the energy functional

$$\mathcal{E}(\psi) = \int_{\mathbb{R}^d} \mathrm{d}x \, \Big(\big|\nabla_x \psi(x)\big|^2 + V(x) \, |\psi(x)|^2 \Big)$$

under the constraint

$$\mathcal{J}(\psi) = \int_{\mathbb{R}^d} \mathrm{d}x \, |\psi(x)|^2 - 1 = 0.$$

Hand in home work on: Thursday, 27 March 2014, before class