Functionals

Homework Problems

63. The Hessian of an action functional (10 points)

Consider the action functional

$$S(q) := \int_0^T \mathrm{d}t\, L\big(q(t), \dot{q}(t)\big)$$

associated to the Lagrange function $L \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d)$ on

$$\mathcal{D}(x_0, x_1) := \Big\{ q \in \mathcal{C}^1\big([0,T], \mathbb{R}^d\big) \;\big|\; q(0) = x_0,\ q(T) = x_1 \Big\}.$$

(i) Argue why it suffices to consider tangent vectors of the form $h \in \mathcal{D}(0,0)$.

(ii) Consider for simplicity the case $d = 1$. Compute the Hessian

$$\big\langle h, \big(\mathrm{d}^2\mathcal{E}(q)\big)k\big\rangle = \frac{\partial^2}{\partial s\,\partial r} S\big(q + sh + rk\big)\Big|_{s=0=r}$$

in terms of L where $h, k \in \mathcal{D}(0,0)$. Find an expression which is independent of $\dot{h}$.

Solution:

(i) First of all, for any $q \in \mathcal{D}(x_0, x_1)$ and $h \in \mathcal{D}(0,0)$ the trajectory $q_s := q + s\,h$ is an element of $\mathcal{D}(x_0, x_1)$ [1]: evidently, $q_s \in \mathcal{C}^2\big([0,T], \mathbb{R}^d\big)$ [1] and $q_s(0) = x_0$ as well as $q_s(T) = x_1$ [1]. Thus, $\partial_s q_s(t)|_{s=0} = h(t) \in \mathbb{R}^d$ is a tangent vector at $q(t)$ [1]. Any tangent vector can be written in this fashion, because any tangent vector can be seen as an element of $\mathbb{R}^d$ [1].

(ii)

$$\begin{aligned}
\big\langle h, \big(\mathrm{d}^2\mathcal{E}(q)\big)k\big\rangle &\overset{[1]}{=} \int_0^T \mathrm{d}t\ \frac{\partial^2}{\partial s\,\partial r} L\Big(q(t) + s\,h(t) + r\,k(t)\,,\,\dot{q}(t) + s\,\dot{h}(t) + r\,\dot{k}(t)\Big)\Big|_{s=0=r} \\
&\overset{[1]}{=} \int_0^T \mathrm{d}t\ \frac{\partial}{\partial r}\Big(\partial_x L\big(q(t) + r\,k(t)\,,\,\dot{q}(t) + r\,\dot{k}(t)\big)\,h(t) + \\
&\qquad\qquad + \partial_v L\big(q(t) + r\,k(t)\,,\,\dot{q}(t) + r\,\dot{k}(t)\big)\,\dot{h}(t)\Big)\Big|_{r=0} \\
&\overset{[1]}{=} \int_0^T \mathrm{d}t\ \frac{\partial}{\partial r}\Big(\partial_x L\big(q(t) + r\,k(t)\,,\,\dot{q}(t) + r\,\dot{k}(t)\big) + \\
&\qquad\qquad - \frac{\mathrm{d}}{\mathrm{d}t}\partial_v L\big(q(t) + r\,k(t)\,,\,\dot{q}(t) + r\,\dot{k}(t)\big)\Big)\ h(t)\Big|_{r=0} \\
&\overset{[2]}{=} \int_0^T \mathrm{d}t\ \bigg(k(t)\,\Big(\partial_x^2 L\big(q(t), \dot{q}(t)\big) - \frac{\mathrm{d}}{\mathrm{d}t}\partial_x\,\partial_v L\big(q(t), \dot{q}(t)\big)\Big) + \\
&\qquad\qquad + \dot{k}(t)\,\Big(\partial_v\,\partial_x L\big(q(t), \dot{q}(t)\big) - \frac{\mathrm{d}}{\mathrm{d}t}\partial_v^2 L\big(q(t), \dot{q}(t)\big)\Big)\bigg)\ h(t)
\end{aligned}$$

64. Taylor expansion of functionals (11 points)

Suppose $\mathcal{E}$ is *twice* Gâteaux differentiable on Ω where Ω is an convex subset of a Banach space $\mathcal{X}$.

(i) Show that $\mathcal{E}$ has a Taylor expansion to first order, i. e. for all $x, y \in \Omega$ there exists $\theta \in [0,1]$ such that $\mathcal{E}(x+y) = \mathcal{E}(x) + \big(\mathrm{d}\mathcal{E}(x)\big)(y) + \Big\langle y, \big(\mathrm{d}^2\mathcal{E}(x+\theta\, y)\big)y \Big\rangle$.

(ii) Show that the remainder $R(x,y) = \mathcal{E}(x+y) - \mathcal{E}(x) - \big(\mathrm{d}\mathcal{E}(x)\big)(y)$ is $o(\lVert y \rVert)$.

Solution:

(i) We define $f(s) := \mathcal{E}(x+sy)$ [1]. Evidently, $f \in \mathcal{C}^2$ because $\mathcal{E}$ is twice Gâteaux differentiable. Hence, we may Taylor-expand the scalar function f [1],

$$\begin{aligned} \mathcal{E}(x+sy) = f(s) &\overset{[1]}{=} f(0) + f'(0)\, s + \tfrac{1}{2} f''(\theta)\, s^2 \\ &\overset{[2]}{=} \mathcal{E}(x) + s\,\big(\mathrm{d}\mathcal{E}(x)\big)y + s^2\, \tfrac{1}{2}\big\langle y, \big(\mathrm{d}^2\mathcal{E}(x+\theta\, y)\big)y \big\rangle. \end{aligned}$$

Here, we have used that the remainder of a Taylor series can be expressed as

$$\frac{f^{(k+1)}(\theta)}{(k+1)!}\, s^{k+1}$$

where $\theta \in [0,1]$ depends on s, and in our case $k = 1$ [1]. Plugging in $s = 1$ yields the claim [1].

(ii) This follows just like in the case of the Taylor expansion on $\mathbb{R}$: the continuity of f'' and $\theta \in [0,1]$ imply

$$r(s) \overset{[1]}{=} f(s) - f(0) - s\, f'(0) \overset{[1]}{=} \mathcal{E}(x+sy) - \mathcal{E}(x) - \big(\mathrm{d}\mathcal{E}(x)\big)(sy) \overset{[1]}{=} o(s),$$

and consequently $R(x,y) = o(\lVert y \rVert)$ [1].

65. Hopf bifurcation (11 points)

Consider the following system of ODEs

$$\dot{r} = f(\mu, r) := r\big(\mu - r^2\big)$$
$$\dot{\theta} = -1$$

in two dimensions which are expressed in polar coordinates ($r \geq 0$ being the radius and θ the angle variable). $\mu \in \mathbb{R}$ is the external parameter. We focus on the equation for $\dot{r}$.

(i) Find the fixed points of the vector field for $\dot{r}$. Discuss all cases for the various values of μ.

(ii) Discuss the stability of the fixed points depending on the values of μ. Sketch a phase portrait for each of the cases.

(iii) Identify the bifurcation point $(\mu_{\text{bi}}, r_{\text{bi}})$. Verify that at the bifurcation point $\partial_r f(\mu_{\text{bi}}, r_{\text{bi}}) = 0$.

Solution:

(i) We have to distinguish the cases $\mu \leq 0$ and $\mu > 0$.

$\mu \leq 0$: Only $r_c = 0$ is a fixed point [1].

$\mu > 0$: $r_c = 0$ [1] and $r_c = \sqrt{\mu}$ [1] are the fixed points.

(ii) $\mu \leq 0$: Given that for $r > 0$ we have $\dot{r} < 0$, the fixed point is a stable focus point [1].

$\mu > 0$: We have to distinguish the cases $r < \sqrt{\mu}$ where $\dot{r} > 0$ and $r > \sqrt{\mu}$ where $\dot{r} < 0$ [1]. In both cases $r(t) \to \sqrt{\mu}$ either from the inside ($r < \sqrt{\mu}$) [1] or from the outside ($r > \sqrt{\mu}$) [1]. Thus, $r_c = 0$ is an unstable fixed point [1] while $r_c = \sqrt{\mu}$ is a stable fixed point [1].

(iii) The bifurcation point is $(0, 0)$ [1]. There, the derivative of f with respect to r vanishes,

$$\partial_r f(0,0) = \partial_r\big(-r^3\big)\big|_{r=0} = -3r^2\big|_{r=0} \overset{[1]}{=} 0.$$