UNIVERSITY OF TORONTO

Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Problem Sheet 21
(2014.03.27)

Functionals

This homework sheet is purely for extra credit.

Homework Problems

63. The Hessian of an action functional

Consider the action functional

$$S(q) := \int_0^T \mathrm{d}t \, L\big(q(t), \dot{q}(t)\big)$$

associated to the Lagrange function $L \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}^d)$ on

$$\mathcal{D}(x_0, x_1) := \Big\{ q \in \mathcal{C}^1\big([0,T], \mathbb{R}^d\big) \;\big|\; q(0) = x_0, \; q(T) = x_1 \Big\}.$$

(i) Argue why it suffices to consider tangent vectors of the form $h \in \mathcal{D}(0,0)$.

(ii) Consider for simplicity the case $d = 1$. Compute the Hessian

$$\big\langle h, \big(\mathrm{d}^2\mathcal{E}(q)\big)k \big\rangle = \frac{\partial^2}{\partial s \, \partial r} S\big(q + sh + rk\big)\bigg|_{s=0=r}$$

in terms of L where $h, k \in \mathcal{D}(0,0)$. Find an expression which is independent of $\dot{h}$.

64. Taylor expansion of functionals

Suppose $\mathcal{E}$ is *twice* Gâteaux differentiable on Ω where Ω is an convex subset of a Banach space $\mathcal{X}$.

(i) Show that $\mathcal{E}$ has a Taylor expansion to first order, i. e. for all $x, y \in \Omega$ there exists $\theta \in [0,1]$ such that $\mathcal{E}(x+y) = \mathcal{E}(x) + \big(\mathrm{d}\mathcal{E}(x)\big)(y) + \Big\langle y, \big(\mathrm{d}^2\mathcal{E}(x + \theta\, y)\big)y \Big\rangle$.

(ii) Show that the remainder $R(x,y) = \mathcal{E}(x+y) - \mathcal{E}(x) - \big(\mathrm{d}\mathcal{E}(x)\big)(y)$ is $o(\lVert y \rVert)$.

65. Hopf bifurcation

Consider the following system of ODEs

$$\dot{r} = f(\mu, r) := r\big(\mu - r^2\big)$$
$$\dot{\theta} = -1$$

in two dimensions which are expressed in polar coordinates ($r \geq 0$ being the radius and θ the angle variable). $\mu \in \mathbb{R}$ is the external parameter. We focus on the equation for $\dot{r}$.

(i) Find the fixed points of the vector field for $\dot{r}$. Discuss all cases for the various values of μ.

(ii) Discuss the stability of the fixed points depending on the values of μ. Sketch a phase portrait for each of the cases.

(iii) Identify the bifurcation point $(\mu_{\mathrm{bi}}, r_{\mathrm{bi}})$. Verify that at the bifurcation point $\partial_r f(\mu_{\mathrm{bi}}, r_{\mathrm{bi}}) = 0$.

Hand in home work on: Thursday, 3 April 2014, before class