Differential Equations of Mathematical Physics
(APM 351 Y)

2013–2014
Solutions 22
(2014.04.22)

Review for Final Exam

The coordinates of the Final Exam in space-time are

Tuesday, 29 April, 14-17 , in the Central Exams Facility, room EX 310.

The *same rules* as last time apply: *no aids* of any kind are allowed. And please bring a *photo id*.

The final exam covers the material from the whole course, i. e. *all* of the homework assignments and the lecture notes from cover to cover. This review sheet is meant to be **in addition** to the review sheets for Tests 2 and 3!

1 Material covered in Test 1

ODEs

ODEs are the content of Chapter 2, although many of the applications come later in Chapter 3 (classical mechanics). It serves as the theoretical foundation for much of the discussion of hamiltonian dynamics.

(1) Review the notion of flow associated to an ODE.

(2) Review the Picard-Lindelöf Theorem as standard existence and uniqueness result for ODEs. Make sure to be able to apply it.

Exercise:

(i) Review homework problems 8 and 9.

(ii) Review problem 2 of Test 1.

(3) Study the Grönwall lemma and its implications for the existence and uniqueness theorem. Review its applications (e. g. ε-closeness of flows associated to ε-close vector fields, Proposition 2.2.7, and homework problem 6).

(4) You should be able to make a *stability analysis* of an ODE.

Exercise:

(i) Repeat problem 2 of Test 1.

(ii) Review Chapter 3.6 carefully.

(iii) Repeat homework problem 10.

(5) You should know the solution to inhomogeneous linear ODEs.

Exercise: Review homework problem 7.

Classical mechanics

As one of the three major physical theories (classical mechanics, quantum mechanics and electrodynamics), the chapter on classical mechanics (Chapter 3) is *extremely* important.

(1) Familiarize yourself with the structure of the hamiltonian framework of classical mechanics, i. e. identify states, observables and dynamical equations in both, Heisenberg and Schrödinger picture. In particular, give Hamilton's equations of motion.

(2) You should be able to perform a stability analysis of Hamilton's equations of motion. (For exercises, have a look at the problems mentioned in the ODE section above.)

(3) Find out how magnetic fields are included in the hamiltonian framework (cf. Chapter 3.5).

Exercise:

(i) Show that minimal substitution and using the magnetic symplectic form are two equivalent ways of including magnetic fields in hamiltonian mechanics.

(ii) Repeat problem 10.

Banach and Hilbert spaces

Chapter 4 which covers Banach and Hilbert spaces is a supplementary chapter, and builds the foundation for much of what we do later on. Make sure to study the product ansatz (aka separation of variables)!

(1) You should know the definition of Banach and Hilbert space.

(2) Enumerate some of the fundamental properties of $L^p(\mathbb{R}^d)$ and $\ell^p(\mathbb{Z}^d)$ spaces (Chapter 4.1.2).

(3) Have a look at fundamental examples for Hilbert spaces and how that ties in with the theory of quantum mechanics.

(4) Understand the *separation of variables ansatz/product ansatz* for solving PDEs with boundary conditions (Chapter 4.1.3).

Exercise:

(i) Review problem 1 of Test 1.

(ii) Repeat the example of the wave equation with boundary conditions on pp. 45–46 in the lecture notes.

2 Variational calculus

Chapter 10 has not been covered by any of the term tests, so please do not forget it in your preparation for the final.

(1) Recap the main points of Chapter 10: What is a functional? How do you find extrema to a functional? What is the difference between a functional and a function (in the ordinary sense)?

(2) You should be able to compute the Gâteaux derivative of a functional and relate it to the Euler-Lagrange equations.

Exercise:

(i) Review Chapter 10.1.4.

(ii) Repeat homework problems 63 and 64.

(iii) Derive the Ginzburg-Landau equations (cf. pp. 175–176 in the lecture notes).

(3) Understand how extrema of functionals are found in the presence of constraints. Refresh your memory on the technique of Lagrange multipliers, and how constraints can be implemented by a suitable choice of coordinates.

Exercise: Have a look at the Example on pp. 173–174 in the lecture notes.

(4) Rehash how functionals can be Taylor expanded (cf. homework problem 64). What consequences does that have for the uniqueness of minima in view of Chapter 10.2?