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Semiclassical Dynamics of a Particle with Spin and Application to Jaynes-Cummings-type Models

1 Setup

semiclassical limit in the sense of Egorov

$$\hat{H}^\varepsilon = \hat{H}_0 + \varepsilon \hat{H}_1 = \hat{h}_0 \otimes \mathrm{id}_{\mathbb{C}^N} + \varepsilon \hat{h}_1 = \hat{h}_0 \otimes \mathrm{id}_{\mathbb{C}^N} + \varepsilon \sum_{j=1}^{3} \hat{\mathfrak{h}}_j \otimes \sigma_j + \cancel{\hat{\mathfrak{h}}_0 \otimes \mathrm{id}}$$

selfadjoint operator on $L^2(\mathbb{R}^d, \mathbb{C}^N)$, $\hat{h}_0, \hat{\mathfrak{h}}_j$ operators on $L^2(\mathbb{R}^d)$

observable $\hat{A} = \hat{A}^*$

<u>Theorem</u> (Egorov-type)

$$\left\| e^{+i\hat{H}^\varepsilon \frac{t}{\varepsilon}} \hat{A} e^{-i\hat{H}^\varepsilon \frac{t}{\varepsilon}} - \widehat{A \circ \Phi_t^0} \right\| = \mathcal{O}(\varepsilon)$$

for times $t \in [-T, +T]$ (uniformly in ε)

where Φ_t^0 is the flow associated to

$$\dot{q} = +\nabla_p h_0$$
$$\dot{p} = -\nabla_q h_0$$

$N=1$: standard (e.g. [Robert])

$N \geq 2$: [Bolte, Keppeler 1998; Keppeler (PhD); Bolte, Glaser 2004, 2005]

<u>our work</u>

① improve error from $\mathcal{O}(\varepsilon)$ to $\mathcal{O}(\varepsilon^{2})$ [~~$0 < \vartheta < 1$~~] ⤳ includes back-reaction of spin onto translational dyn

② special case $d=1$, $N=2$, $h_0(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2)$, $\mathfrak{h}_j(q,p) = \alpha_j + \beta_j q + \gamma_j p$

(⤳ covers Jaynes-Cummings, Rabi)

long times: $t = \mathcal{O}(1/\varepsilon^\gamma)$, Egorov with error $\mathcal{O}(\varepsilon^{1-\vartheta})$, $\vartheta = \vartheta(\gamma) \in [0,1)$

hard: cannot use Grönwall lemma to obtain [~~estimates~~] bounds on the derivatives of the flow!

in what follows: assume $\hat{H}^\varepsilon$ is as in ② for simplicity

2 Stratonovich-Weyl calculus for spin

	Weyl system	Poisson
ΨDC on $\mathbb{R}^{2d}$	$W(q,p) = e^{-i(p\cdot\hat{x} - q\cdot(-i\varepsilon\nabla_x))}$	$\{F,G\}_{\mathbb{R}^d} = \nabla_p F\cdot\nabla_q G - \nabla_q F\cdot\nabla_p G$
ΨDC on $\mathbb{S}^2$	$\Delta(n) = \frac{1}{2}(\mathrm{id}_{\mathbb{C}^2} + \sqrt{3}\, n\cdot\sigma)$	$\{f,g\}_{\mathbb{S}^2} = -\frac{1}{\sqrt{3}/2}(\nabla_n f \wedge \nabla_n g)\cdot n$

quantization

$\mathrm{Op}_{\mathbb{R}^d}(F) := \frac{1}{(2\pi)^d}\int dq\,dp \cdot (\mathfrak{F}F)(q,p)\, W(q,p)$

$\mathrm{Op}_{\mathbb{S}^2}(f) = \frac{1}{4\pi}\int dn\, f(n)\,\Delta(n)$

Moyal product

$F \#_{\mathbb{R}^d} G = FG - \varepsilon \frac{i}{2}\{F,G\}_{\mathbb{R}^d} + \mathcal{O}(\varepsilon^2)$

$f \#_{\mathbb{S}^2} g$

commutator — $F \vee G$ scalar

$[F,G]_{\#_{\mathbb{R}^d}} \overset{\circledast}{=} -\varepsilon i\{F,G\}_{\mathbb{R}^d} + \mathcal{O}(\varepsilon^3)$

$[f,g]_{\#_{\mathbb{S}^2}} \overset{\circledast}{=} -i\{f,g\}_{\mathbb{S}^2}, \quad f,g \in \mathcal{C}_1(\mathbb{S}^2)$

Wigner transform

<u>important difference</u> $\mathrm{Op}_{\mathbb{S}^2}$ is <u>not</u> injective: $\dim \mathcal{C}^\infty(\mathbb{S}^2) = \infty$ (classical), $\dim \mathcal{B}(\mathbb{C}^2) = 4$ (quantum)

$\rightsquigarrow \mathrm{Op}_{\mathbb{S}^2} : \mathcal{C}_1(\mathbb{S}^2) \longrightarrow \mathcal{B}(\mathbb{C}^2)$ is bijective ($\mathcal{C}_1(\mathbb{S}^2) :=$ first-order polynomials in n)

space of relevant observables: $\mathcal{C}^\infty(\mathbb{R}^{2d}) \otimes \mathcal{C}_1(\mathbb{S}^2)$

symplectic form $\longleftrightarrow$ Poisson bracket on $\Sigma := \mathbb{R}^{2d} \times \mathbb{S}^2$

$$\{f,g\}_\Sigma := \{f,g\}_{\mathbb{R}^d} + \frac{1}{\varepsilon}\{f,g\}$$

semiclassical dynamics:

$$\frac{d}{dt} f(t) = \{h^\varepsilon, f(t)\}_\Sigma, \quad f(0) = f \in \mathcal{C}^\infty(\mathbb{R}^{2d}) \otimes \mathcal{C}_1(\mathbb{S}^2)$$

$\Longleftrightarrow f(t) = f \circ \Phi^\varepsilon_t$, Φ^ε_t flow on Σ associated to

$$\dot{q} = +\nabla_p h^\varepsilon = +\nabla_p h_0 + \varepsilon\nabla_p h_1$$

$$\dot{p} = -\nabla_q h^\varepsilon = -\nabla_q h_0 - \varepsilon\nabla_q h_1$$

contain spin

$$\dot{n} = 2\, h \wedge n$$

3 High-precision semiclassics

<u>Theorem</u> (Gat-L.-Teufel, 2012)
$f \in \mathcal{C}^\infty_{\underline{b}}(\mathbb{R}^{2d}) \otimes \mathcal{C}_1(\mathbb{S}^2)$, standard assumptions on h (à la Robert, second-order + derivatives of h_0 bounded, first-order + derivatives on h_1 bounded)
ⓘ If f is independent of n, then Egorov holds with $\mathcal{O}(\varepsilon^{\underline{2}})$ error.
ⓘⓘ If f depends non-trivially on n, then Egorov holds with $\mathcal{O}(\varepsilon^{\underline{1}})$, and in general, one cannot do better.

<u>Proof</u> (Sketch)
ⓘ Start with usual Duhamel argument, obtaining

$$[\hat{H}^\varepsilon, \widehat{F(t)}] - \cancel{\mathrm{Op}_\Sigma(f\circ\Phi}\ \frac{d}{dt}\mathrm{Op}_\Sigma(f\circ\Phi^\varepsilon_t)$$

$+ \mathcal{O}(\varepsilon^2)$ term

- Use Grönwall lemma for estimates on $\frac{d}{dt} f\circ\tilde{\Phi}^\varepsilon_t$ and its (q,p) derivatives ⇝ restriction to $\mathcal{O}(1)$ times enters here
- Feed that into Calderon–Vaillancourt theorem (needs ~~2d+1~~ control of derivatives up to $(2d+1)$th order
- replace Φ^ε_t with $\tilde{\Phi}^\varepsilon_t = (\tilde{\phi}^\varepsilon_t, \mathrm{id}_{\mathbb{S}^2})$ where $\tilde{\phi}^\varepsilon_t$ is flow on $\mathbb{R}^{2d}$ which solves

$$\dot{q} = +\nabla_p h_0 + \varepsilon \sum_{j=1}^{3} \nabla_p\big(h_{1j}\, n_j^{(0)}(t)\big)$$
$$\dot{p} = -\nabla_q h_0 - \varepsilon \sum_{j=1}^{3} \nabla_q\big(h_{1j}\, n_j^{(0)}(t)\big)$$

where $n^{(0)}(t) := n\circ\Phi^0_t$
$\Rightarrow n\circ\Phi^\varepsilon_t - n\circ\Phi^0_t = \mathcal{O}(\varepsilon)$ (Grönwall)
$\Rightarrow$ since n-dependence appears at first order for (q,p):

$$f\circ\Phi^\varepsilon_t - f\circ\tilde{\Phi}^\varepsilon_t = \mathcal{O}(\varepsilon^2)$$

ⓘⓘ Arguments fail since if f depends non-trivially on n:

$$\cancel{f\circ\Phi^\varepsilon_t - f\circ\tilde{\Phi}^\varepsilon_t = \mathcal{O}(\varepsilon^1)}$$

and $f\circ\Phi^\varepsilon_t \in \mathcal{C}^\infty_b(\mathbb{R}^{2d})\otimes\mathcal{C}_1(\mathbb{S}^2)$ <u>only up to $\mathcal{O}(\varepsilon)$</u> $\Rightarrow$ simple semiclassical equations of motion false! (projected dynamics)

4 Long-time semiclassics

- assume $h^\varepsilon = h_0 + \varepsilon h_1$ is of "Rabi-type": $h_0(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2)$
  $h_{1j}(q,p) = \alpha_j + \beta_j q + \gamma_j p$

<u>Theorem</u> (Gat, L., Teufel 2012)

Assume $f \in \mathcal{C}^\infty_{\underline{\underline{c}}}(\mathbb{R}^2) \otimes \mathcal{C}_1(S^2)$ and $T < \infty$.

(i) $\forall \gamma \in [0, 1/4)$ $\exists \varepsilon_0 > 0$ such that

$$\sup_{|t| \in [0, T/\varepsilon^{\gamma}]} \left\| e^{+i\hat{H}^\varepsilon \frac{t}{\varepsilon}} \mathrm{Op}_\Sigma(f)\, e^{-i\hat{H}^\varepsilon \frac{t}{\varepsilon}} - \mathrm{Op}_\Sigma(f \circ \Phi^\varepsilon_t) \right\| = \mathcal{O}(\varepsilon^{1-4\gamma})$$

holds uniformly in $\varepsilon \in [0, \varepsilon_0)$

(ii) If ~~[f is]~~ in addition f is <u>independent of n</u>, then $\forall \gamma \in [0, 1/2)$ $\exists \varepsilon_0 > 0$:

$$\sup_{|t| \in [0, T/\varepsilon^{\gamma}]} \left\| e^{+i\hat{H}^\varepsilon \frac{t}{\varepsilon}} \mathrm{Op}_\Sigma(f)\, e^{-i\hat{H}^\varepsilon \frac{t}{\varepsilon}} - \mathrm{Op}_\Sigma(f \circ \Phi^\varepsilon_t) \right\| = \mathcal{O}(\varepsilon^{\frac{3}{2}-3\gamma})$$

holds uniformly in $\varepsilon \in (0, \varepsilon_0)$.

<u>Proof</u> (Sketch)

- h^ε quadratic $\Rightarrow$ asymptotic expansion of $[\mathrm{Op}_{S^2}(h^\varepsilon), \mathrm{Op}_{S^2}(f)]_{\#_R}$ terminates after $\mathcal{O}(\varepsilon^2)$ term
- from Calderón-Vaillancourt for $d=1$: need control over $2 \cdot 1 + 1 = 3$ derivatives of flow Φ^ε_t
- instead of Grönwall lemma argument: estimate derivatives by hand using (1) h_0 is harmonic oscillator and (2) h_{1j} are linear polynomials. □