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Grade

Last name	First name
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I . . . .
First correction

II . . . .
Second correction

University of Toronto
Department of Mathematics

Test 1
Differential Equations of Mathematical Physics
(APM 351 Y)

Max Lein

5 November 2013, 17:10–18:50, Galbraith Building, GB 119

Room: . . . . Row: . . . . Seat: . . . .

Remarks:

Please verify the completeness of the exam: **4** problems

Time allotted: **90** minutes

Allowed aids: **none**

Only to be filled out by the instructor/TA:

Left room from . . . . to . . . .

Handed in early at . . . .

Remarks:

Solution

1. **The heat equation (18 points)**

Consider the heat equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x) \tag{1}$$

on the interval $[0,L]$ with *Neumann boundary conditions*

$$\partial_x u(t,0) = 0 = \partial_x u(t,L)\,.$$

(i) Derive the solution using separation of variables.

(ii) Compute $\lim_{t\to\infty} u(t,x)$ for any solution of (1), *assuming* you can interchange limit and sum.

(iii) Give a necessary condition on the coefficients which allows you to interchange limit and sum in (ii). Justify your answer.

(iv) Solve (1) for the initial condition $u(0,x) = \big(\cos \frac{\pi}{L} x\big)^2$.

Solution:

(i) We write $u(t,x) = \tau(t)\,\xi(x)$ [1] and plug that ansatz into (1),

$$\begin{aligned} \dot\tau(t)\,\xi(x) &= \tau(t)\,\xi''(x)\,, \\ \Rightarrow\quad \frac{\dot\tau}{\tau} &= \frac{\xi''}{\xi} = \lambda \in \mathbb{C}\,. \qquad [1] \end{aligned}$$

To determine the allowed values of $\lambda \in \mathbb{C}$, we solve the equation for ξ and plug in the Neumann boundary conditions: the above equation yields a harmonic oscillator equation for ξ,

$$\xi'' - \lambda\xi = 0\,, \qquad [1]$$

whose solution is

$$\xi(x) = a_1\,\mathrm{e}^{+x\sqrt{\lambda}} + a_2\,\mathrm{e}^{-x\sqrt{\lambda}}\,. \qquad [1]$$

The derivative needs to satisfy the boundary condition:

$$\begin{aligned} \xi'(x) &= \sqrt{\lambda}\,a_1\,\mathrm{e}^{+x\sqrt{\lambda}} - \sqrt{\lambda}\,a_2\,\mathrm{e}^{-x\sqrt{\lambda}} \\ \xi'(0) &= \sqrt{\lambda}\,\big(a_1 - a_2\big) \overset{!}{=} 0 \quad\Rightarrow\quad a_1 = a_2 \qquad [1] \\ \xi'(L) &= \sqrt{\lambda}\,a_1\big(\mathrm{e}^{+L\sqrt{\lambda}} - \mathrm{e}^{-L\sqrt{\lambda}}\big) \overset{!}{=} 0 \qquad [1] \end{aligned}$$

The latter condition is equivalent to $\mathrm{e}^{2L\sqrt{\lambda}} = 1$ and implies

$$2L\sqrt{\lambda} = \mathrm{i}\,2\pi n\,, \qquad n \in \mathbb{Z}\,. \qquad [1]$$

Hence only

$$\lambda = -\frac{\pi^2}{L^2}\,n^2$$

are allowed for some integer $n = 0, 1, \ldots$, and the solutions are of the form

$$\xi_n(x) = \tfrac{1}{2}a(n)\big(\mathrm{e}^{+\mathrm{i}n\frac{\pi}{L}x} + \mathrm{e}^{-\mathrm{i}n\frac{\pi}{L}x}\big) = a(n)\,\cos n\tfrac{\pi}{L}x\,. \qquad [1]$$

Thus, the associated solutions $\tau_n(t) = \mathrm{e}^{-n^2 \frac{\pi^2}{L^2} t}$ solve

$$\dot{\tau}_n(t) = -n^2 \frac{\pi^2}{L^2} \tau_n(t) , \qquad \tau_n(0) = 1 , \qquad [1]$$

and any solution to (1) is of the form

$$u(t,x) = \sum_{n=0}^{\infty} a(n)\, \mathrm{e}^{-n^2 \frac{\pi^2}{L^2} t} \cos n \tfrac{\pi}{L} x . \qquad [1]$$

(ii) If we can interchange $\lim_{t \to \infty}$ and the sum over n, we obtain

$$\begin{aligned} \lim_{t\to\infty} u(t,x) &\overset{[1]}{=} \lim_{t\to\infty} \sum_{n=0}^{\infty} a(n)\, \mathrm{e}^{-n^2 \frac{\pi^2}{L^2} t} \cos n \tfrac{\pi}{L} x \\ &\overset{[1]}{=} \sum_{n=0}^{\infty} \lim_{t\to\infty} \Big(a(n)\, \mathrm{e}^{-n^2 \frac{\pi^2}{L^2} t} \cos n \tfrac{\pi}{L} x \Big) \\ &\overset{[1]}{=} a(0) . \end{aligned}$$

(iii) A sufficient condition to allow the interchange of limit and sum is the absolute summability of the coefficients $a(n)$,

$$\sum_{n=0}^{\infty} \big|a(n)\big| < \infty . \qquad [1]$$

Because then each of the coefficients satisfies $\mathrm{e}^{-n^2 \frac{\pi^2}{L^2} t} \big|a(n)\big| \leq \big|a(n)\big|$, solution exists pointwise for any t and x,

$$\begin{aligned} \big\|u(t,x)\big\| &\leq \sum_{n=0}^{\infty} \mathrm{e}^{-n^2 \frac{\pi^2}{L^2} t} \big|a(n)\big| \big|\cos n \tfrac{\pi}{L} x\big| \\ &\leq \sum_{n=0}^{\infty} \big|a(n)\big| < \infty , \qquad [1] \end{aligned}$$

and thus, by dominated convergence for sums, we may interchange limits and sums [1].

(iv) The square of the cos can be written as a double-angle cos,

$$\big(\cos \tfrac{\pi}{L} x\big)^2 = \tfrac{1}{2} \big(1 + \cos 2 \tfrac{\pi}{L} x\big) . \qquad [1]$$

This means only two coefficents are non-zero, and we obtain as solution

$$u(t,x) = \frac{1}{2} \Big(1 + \mathrm{e}^{-4 \frac{\pi^2}{L^2} t} \cos 2 \tfrac{\pi}{L} x \Big) . \qquad [1]$$

2. **Classical mechanics (24 points)**

Consider Hamilton's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = X_H := \begin{pmatrix} +\partial_p H \\ -\partial_q H \end{pmatrix} \tag{2}$$

on $\mathbb{R}^2$ associated to the Hamilton function

$$H(p,q) = \sqrt{m^2+p^2} + V(q)\,.$$

(i) Assume $V \in \mathcal{C}^2(\mathbb{R})$. Find the fixed points of the Hamiltonian vector field X_H and characterize the stability of each fixed point in terms of V (stable or unstable, elliptic or hyperbolic).

(ii) Show that for $V(q) = q - 2\,\log\bigl(1+q^2\bigr)$ the Hamiltonian flow Φ exists for all $t \in \mathbb{R}$.

(iii) For this potential $V(q) = q - 2\,\log\bigl(1+q^2\bigr)$, find all fixed points and characterize their stability (stable or unstable, elliptic or hyperbolic).

Solution:

(i) Fixed points satisfy the condition

$$X_H(q_c,p_c) = \begin{pmatrix} \frac{p}{\sqrt{m^2+p^2}} \\ -V'(q) \end{pmatrix} \overset{!}{=} 0 \qquad [1]$$

and thus, $p_c = 0$ and $V'(q_c) = 0$ [1].

If we evaluate the differential of the vector field

$$DX_H(q,p) \overset{[1]}{=} \begin{pmatrix} 0 & m^2\,\bigl(m^2+p^2\bigr)^{-3/2} \\ -V''(q) & 0 \end{pmatrix}$$

at a critical point $(q_c,0)$ and compute the eigenvalues [1], we arrive at

$$\det\bigl(\lambda\,\mathrm{id} - DX_H(q_c,0)\bigr) = \det\begin{pmatrix} \lambda & -m^{-1} \\ +V''(q) & \lambda \end{pmatrix} \overset{[1]}{=} \lambda^2 + m^{-1}\,V''(q_c)\,.$$

If $V''(q_c) > 0$, then the two eigenvalues are purely imaginary [1], and thus, the fixed point $(q_c,0)$ is stable and elliptic [1].

If $V''(q_c) < 0$, then the two eigenvalues are real, one positive, one negative [1], and thus, the fixed point $(q_c,0)$ is unstable and hyperbolic [1].

(ii) We have to check whether the Hamiltonian vector field

$$X_H(q,p) = \begin{pmatrix} +\partial_p H(q,p) \\ -\partial_q H(q,p) \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} \frac{p}{\sqrt{m^2+p^2}} \\ -V'(q) \end{pmatrix}$$

is Lipschitz on all of $\mathbb{R}^2$ [1]. We can treat both entries separately, because the first component only depends on p and the second depends only on q. The *second* derivative of the kinetic energy is bounded,

$$\begin{aligned}\partial_p^2\Bigl(\sqrt{m^2+p^2}\Bigr) &= \partial_p\left(\frac{p}{\sqrt{m^2+p^2}}\right) \overset{[1]}{=} \bigl(m^2+p^2\bigr)^{-1/2} + p\,\bigl(-\tfrac{1}{2}\bigr)\,2p\,\bigl(m^2+p^2\bigr)^{-3/2} \\ &= \bigl(m^2+p^2\bigr)^{-3/2}\,\bigl(m^2+p^2-p^2\bigr) \overset{[1]}{=} m^2\,\bigl(m^2+p^2\bigr)^{-3/2}\,,\end{aligned}$$

and thus, $\partial_q\left(\sqrt{m^2+p^2}\right)$ is Lipschitz [1]. Moreover, we have to check whether V' is Lipschitz, and a sufficient condition for V' to be Lipschitz is the boundedness of V'': the power of the numerator is smaller than the power of the strictly positive denominator [1],

$$\begin{aligned}\partial_q^2\big(q - 2\,\log(1+q^2)\big) &= \partial_q\left(1 - \frac{4q}{1+q^2}\right) = -\frac{4}{1+q^2} + \frac{8q^2}{(1+q^2)^2}\\ &= \frac{8q^2 - 4(1+q^2)}{(1+q^2)^2} \overset{[1]}{=} \frac{4q^2-4}{(1+q^2)^2}\,,\end{aligned}$$

and hence, V' is Lipschitz.

Thus, the Hamiltonian vector field X_H is Lipschitz for all of $(q,p) \in \mathbb{R}^2$ [1], and by the Picard-Lindelöf theorem, the Hamiltonian flow exists for all $t \in \mathbb{R}$ [1].

(iii) Fixed points need to satisfy

$$X_H(q,p) = \begin{pmatrix} \frac{p}{\sqrt{m^2+p^2}} \\ -1 + \frac{4q}{1+q^2} \end{pmatrix} = 0\,,$$

and we deduce $p = 0$ and q needs to be a critical point of $q - 2\,\log(1+q^2)$:

$$-1 + \frac{4q}{1+q^2} \overset{!}{=} 0 \iff 1 + q^2 - 4q = q^2 - 4q + 1 \overset{!}{=} 0 \qquad [1]$$

The zeros of this quadratic polynomial are $q_\pm = 2 \pm \sqrt{3}$. Hence, the fixed points are $\big(2\pm\sqrt{3}, 0\big)$ [1].

Since the denominator of the second derivative is always positive, we only need to concern ourselves with the sign of the numerator: $V''\big(2+\sqrt{3}\big)$ is negative,

$$4\left(1 - \big(2+\sqrt{3}\big)^2\right) = 4\left(1 - 4 - 4\sqrt{3} - 3\right) = -4\left(6 + 4\sqrt{3}\right) < 0\,, \qquad [1]$$

while $V''\big(2-\sqrt{3}\big)$ is positive, because $2 - \sqrt{3} < 1$, and thus

$$\underbrace{4\left(1 - \underbrace{\big(2-\sqrt{3}\big)^2}_{<1}\right)}_{>0} > 0\,. \qquad [1]$$

This means, $\big(2+\sqrt{3}, 0\big)$ is a stable, elliptic fixed point [1] while $\big(2-\sqrt{3}, 0\big)$ is a unstable and hyperbolic [1].

3. **The Schrödinger equation for spin (17 points)**

For $b > 0$ consider the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H\psi(t) := \begin{pmatrix} 0 & -\mathrm{i}\,b \\ +\mathrm{i}\,b & 0 \end{pmatrix}\psi(t)\,, \qquad \psi(0) \in \mathbb{C}^2\,, \tag{3}$$

on the Hilbert space $\mathbb{C}^2$ with scalar product $\langle\psi, \varphi\rangle := \sum_{j=1,2} \overline{\psi_j}\,\varphi_j$.

(i) Compute the flow Φ. (Hint: Compute the powers of H explicitly.)

(ii) Elaborate in what sense Φ_t exists.

(iii) Solve the initial value problem for $\psi(0) = (1, 0)$.

(iv) Show that Φ_t is unitary.

Solution:

(i) First of all, the flow is given by $\Phi_t := \mathrm{e}^{-\mathrm{i}tH}$ [1]. To compute the matrix exponential explicitly, we note that

$$\begin{aligned} H^0 &= \mathrm{id}_{\mathbb{C}^2}\,, \\ H^1 &= H\,, \\ H^2 &= \begin{pmatrix} 0 & -\mathrm{i}\,b \\ +\mathrm{i}\,b & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i}\,b \\ +\mathrm{i}\,b & 0 \end{pmatrix} = \begin{pmatrix} -b^2\,\mathrm{i}^2 & 0 \\ 0 & -b^2\,\mathrm{i}^2 \end{pmatrix} = b^2\,\mathrm{id}_{\mathbb{C}^2}\,, \end{aligned}$$

and hence, $H^{2n} = b^{2n}\,\mathrm{id}_{\mathbb{C}^2}$ and $H^{2n+1} = H^{2n}\,H = b^{2n}\,H$ [1]. Thus, we can split the sum for the matrix exponential and compute it explicitly:

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}tH} &\overset{[1]}{=} \sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!}\,H^n \\ &\overset{[1]}{=} \sum_{n=0}^{\infty}\left(\frac{(-\mathrm{i}t)^{2n}}{(2n)!}\,H^{2n} + \frac{(-\mathrm{i}t)^{2n+1}}{(2n+1)!}\,H^{2n+1}\right) \\ &\overset{[1]}{=} \left(\sum_{n=0}^{\infty}(-1)^n\,\frac{(bt)^{2n}}{(2n)!}\right)\mathrm{id}_{\mathbb{C}^2} - \frac{\mathrm{i}}{b}\left(\sum_{n=0}^{\infty}(-1)^n\,\frac{(bt)^{2n+1}}{(2n+1)!}\right)H \\ &\overset{[1]}{=} \cos(bt)\,\mathrm{id}_{\mathbb{C}^2} - \tfrac{\mathrm{i}}{b}\,\sin(bt)\,H \end{aligned}$$

(ii) $\mathrm{e}^{-\mathrm{i}tH}$ can be seen as the the limit of partial sums $\sum_{n=0}^{N} \frac{(-\mathrm{i}t)^n}{n!}\,H^n$ as $N \to \infty$ which converges in the operator norm $\mathcal{B}(\mathbb{C}^2)$ (or any norm in the space of matrices) [1].

(iii) We use the matrix exponential computed in (i) and

$$\begin{aligned} \psi(t) = \mathrm{e}^{-\mathrm{i}tH}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &\overset{[1]}{=} \left(\cos(bt)\,\mathrm{id}_{\mathbb{C}^2} - \tfrac{\mathrm{i}}{b}\,\sin(bt)\,H\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\overset{[1]}{=} \begin{pmatrix} \cos(bt) \\ 0 \end{pmatrix} - \frac{\mathrm{i}}{b}\,\sin(bt)\begin{pmatrix} 0 & -\mathrm{i}\,b \\ +\mathrm{i}\,b & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(bt) \\ 0 \end{pmatrix} - \frac{\mathrm{i}}{b}\begin{pmatrix} 0 \\ +\mathrm{i} \end{pmatrix} \overset{[1]}{=} \begin{pmatrix} \cos(bt) \\ \sin(b) \end{pmatrix} \end{aligned}$$

Clearly, $\psi(t)$ satisfies the initial value problem: $\psi(0) = \big(\cos(0), \sin(0)\big) = (1, 0)$ [1].

(iv) First of all, since $\Phi_t = \mathrm{e}^{-\mathrm{i}tH}$ satisfies the group property,

$$\mathrm{e}^{-\mathrm{i}t_1 H}\,\mathrm{e}^{-\mathrm{i}t_2 H} = \mathrm{e}^{-\mathrm{i}(t_1+t_2)H}\,,$$

the inverse is $\left(\mathrm{e}^{-\mathrm{i}tH}\right)^{-1} = \mathrm{e}^{+\mathrm{i}tH}$ [1].
On the other hand, we note that H is selfadjoint [1],

$$\begin{aligned} H^* &= \begin{pmatrix} 0 & -\mathrm{i}\,b \\ +\mathrm{i}\,b & 0 \end{pmatrix}^* = \begin{pmatrix} \overline{0} & \overline{+\mathrm{i}\,b} \\ \overline{-\mathrm{i}\,b} & \overline{0} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\mathrm{i}\,b \\ +\mathrm{i}\,b & 0 \end{pmatrix} = H\,, \end{aligned}$$

and hence, we deduce that $\Phi_t = \mathrm{e}^{-\mathrm{i}tH}$ is unitary:

$$\begin{aligned} \left(\mathrm{e}^{-\mathrm{i}tH}\right)^* &\overset{[1]}{=} \left(\sum_{n=0}^{\infty} \frac{(-\mathrm{i}t)^n}{n!}\,H^n\right)^* = \sum_{n=0}^{\infty}\left(\frac{(-\mathrm{i}t)^n}{n!}\,H^n\right)^* \\ &\overset{[1]}{=} \sum_{n=0}^{\infty} \frac{(+\mathrm{i}t)^n}{n!}\,H^{*n} = \sum_{n=0}^{\infty} \frac{(+\mathrm{i}t)^n}{n!}\,H^n \\ &\overset{[1]}{=} \mathrm{e}^{+\mathrm{i}tH} \overset{[1]}{=} \left(\mathrm{e}^{-\mathrm{i}tH}\right)^{-1} \end{aligned}$$

4. **Orthogonal projections (12 points)**

Let P and Q be two orthogonal projections on a Hilbert space $\mathcal{H}$.

(i) Assume in addition $P\,Q = 0$. Show that $P + Q$ is an orthogonal projection.

(ii) Show that either $\|P\| = 0$ or $\|P\| = 1$.

Solution:

(i) The condition $P\,Q = 0$ and the selfadjointness of P and Q also imply

$$Q\,P \overset{[1]}{=} Q^*\,P^* \overset{[1]}{=} \big(PQ\big)^* \overset{[1]}{=} 0\,.$$

Thus, $P + Q$ is a projection,

$$\big(P+Q\big)^2 \overset{[1]}{=} P^2 + \underbrace{P\,Q}_{=0} + \underbrace{Q\,P}_{=0} + Q^2 \overset{[1]}{=} P + Q\,.$$

Since $P^* = P$ and $Q^* = Q$, the sum

$$\big(P+Q\big)^* \overset{[1]}{=} P^* + Q^* \overset{[1]}{=} P + Q$$

is also selfadjoint. Hence, $P + Q$ is an orthogonal projection [1].

(ii) By the properties of the Hilbert adjoint, we deduce

$$\|P\|^2 \overset{[1]}{=} \big\|P^*\,P\big\| \overset{[1]}{=} \big\|P^2\big\| \overset{[1]}{=} \|P\|\ .$$

This equation is only satisfied if either $\|P\| = 0$ or $\|P\| = 1$ [1].