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Grade

Last name

First name

Student id #

Major

Signature

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I . . . .
First correction

II . . . .
Second correction

University of Toronto
Department of Mathematics

Test 1
Differential Equations of Mathematical Physics
(APM 351 Y)

Max Lein

5 November 2013, 17:10–18:50, Galbraith Building, GB 119

Room: . . . . Row: . . . . Seat: . . . .

Remarks:
Please verify the completeness of the exam: **4** problems

Time allotted: **90** minutes

Allowed aids: **none**

Only to be filled out by the instructor/TA:

Left room from . . . . to . . . .

Handed in early at . . . .

Remarks:

1. **The heat equation (18 points)**
Consider the heat equation

$$\partial_t u(t,x) = \partial_x^2 u(t,x) \tag{1}$$

on the interval $[0, L]$ with *Neumann boundary conditions*

$$\partial_x u(t,0) = 0 = \partial_x u(t,L)\,.$$

(i) Derive the solution using separation of variables.

(ii) Compute $\lim_{t\to\infty} u(t,x)$ for any solution of (1), *assuming* you can interchange limit and sum.

(iii) Give a necessary condition on the coefficients which allows you to interchange limit and sum in (ii). Justify your answer.

(iv) Solve (1) for the initial condition $u(0,x) = \left(\cos \frac{\pi}{L} x\right)^2$.

2. **Classical mechanics (24 points)**
Consider Hamilton's equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} q \\ p \end{pmatrix} = X_H := \begin{pmatrix} +\partial_p H \\ -\partial_q H \end{pmatrix} \tag{2}$$

on $\mathbb{R}^2$ associated to the Hamilton function

$$H(p, q) = \sqrt{m^2 + p^2} + V(q) \, .$$

(i) Assume $V \in \mathcal{C}^2(\mathbb{R})$. Find the fixed points of the Hamiltonian vector field X_H and characterize the stability of each fixed point in terms of V (stable or unstable, elliptic or hyperbolic).

(ii) Show that for $V(q) = q - 2 \, \log\bigl(1 + q^2\bigr)$ the Hamiltonian flow Φ exists for all $t \in \mathbb{R}$.

(iii) For this potential $V(q) = q - 2 \, \log\bigl(1 + q^2\bigr)$, find all fixed points and characterize their stability (stable or unstable, elliptic or hyperbolic).

3. **The Schrödinger equation for spin (17 points)**

For $b > 0$ consider the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H\psi(t) := \begin{pmatrix} 0 & -\mathrm{i}\,b \\ +\mathrm{i}\,b & 0 \end{pmatrix} \psi(t)\,, \qquad \psi(0) \in \mathbb{C}^2\,, \tag{3}$$

on the Hilbert space $\mathbb{C}^2$ with scalar product $\langle \psi, \varphi \rangle := \sum_{j=1,2} \overline{\psi_j}\,\varphi_j$.

(i) Compute the flow Φ. (Hint: Compute the powers of H explicitly.)

(ii) Elaborate in what sense Φ_t exists.

(iii) Solve the initial value problem for $\psi(0) = (1, 0)$.

(iv) Show that Φ_t is unitary.

4. **Orthogonal projections (12 points)**
Let P and Q be two orthogonal projections on a Hilbert space $\mathcal{H}$.

(i) Assume in addition $PQ = 0$. Show that $P + Q$ is an orthogonal projection.

(ii) Show that either $\|P\| = 0$ or $\|P\| = 1$.