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University of Toronto
Department of Mathematics

# Solution to Test 2
# Differential Equations of Mathematical Physics
# (APM 351 Y)

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30 January 2014, 09:10–10:50, Sidney Smith Hall, SS 1074

**Remarks:**
Please verify the completeness of the exam: **4** problems

Time allotted: **90** minutes

Allowed aids: **none**

1. **Partial differential equation on $\mathbb{T}^2$ (12 points)**

Consider the PDE

$$\partial_{x_1}^8 u - 2\partial_{x_2}^6 u + 3u = f$$

on $\mathbb{T}^2$ with $f \in \mathcal{C}^2(\mathbb{T}^2)$.

(i) Give the definition of the discrete Fourier transform $\mathcal{F} : L^1(\mathbb{T}^d) \longrightarrow \ell^\infty(\mathbb{Z}^d)$ and its inverse. (You may assume that the sum converges.)

(ii) Find the solution $u$.

(iii) Investigate the smoothness of the solution, i. e. find the largest integer $k$ so that $u \in \mathcal{C}^k(\mathbb{T}^2)$.

**Solution:**

(i)

$$(\mathcal{F}f)(\xi) \overset{[1]}{=} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}\xi\cdot x} \, f(x)$$

$$\big(\mathcal{F}^{-1}\hat{f}\big)(x) \overset{[1]}{=} \sum_{\xi\in\mathbb{Z}^d} \hat{f}(\xi) \, \mathrm{e}^{+\mathrm{i}\xi\cdot x}$$

(ii) Taking the discrete Fourier transform on both sides yields

$$\mathcal{F}\Big(\partial_{x_1}^8 u - 2\partial_{x_2}^6 u + 3u\Big) \overset{[1]}{=} \big(\mathrm{i}^8 \, \xi_1^8 - \mathrm{i}^6 \, 2 \, \xi_2^6 + 3\big)\hat{u}$$

$$\overset{[1]}{=} \big(\xi_1^8 + 2\,\xi_2^6 + 3\big)\hat{u} = \hat{f}.$$

The polynomial $P(\xi) = \xi_1^8 + 2\,\xi_2^6 + 3 > 0$ is positive, and hence $1/P$ is bounded [1]. Moreover, given that $\left|P^{-1}(\xi)\right| \propto |\xi|^{-6}$ for large $|\xi|$, $P^{-1}$ is absolutely summable and $\mathcal{F}^{-1}\big(P^{-1}\big)$ exists [1]. Thus, we obtain the solution after inverse Fourier transform:

$$u(x) \overset{[1]}{=} \mathcal{F}^{-1}\big(P^{-1}\,\hat{f}\,\big) \overset{[1]}{=} (2\pi)^2 \, \mathcal{F}^{-1}\big(P^{-1}\big) * f$$

(iii) The Fourier transform of $f \in \mathcal{C}^2(\mathbb{T}^2)$ decays at least as fast as $|\xi|^{-2}$ for large $|\xi|$ [1] while

$$\left|P(\xi)^{-1}\right| \leq C\,|\xi|^{-6} \qquad [1]$$

holds as $|\xi| \to \infty$, and thus the decay of

$$\left|P^{-1}(\xi)\,\hat{f}(\xi)\right| \leq C'\,|\xi|^{-8} \qquad [1]$$

for large $|\xi|$ implies $u \in \mathcal{C}^5(\mathbb{T}^2)$ ($8 - d - \delta = 6 - \delta$ for some $\delta \in (0,1)$ and $d = 2$) [1].

2. **The heat equation (14 points)**

Consider the one-dimensional heat equation

$$\partial_t u(t) = \tfrac{1}{2}\partial_x^2 u(t) + f(t), \qquad\qquad u(0) = u_0 \in L^1(\mathbb{R}),$$

with inhomogeneity $f$.

(i) Find the solution $u(t)$.

(ii) For the case $f(t,x) = x$ and $u_0 = 0$, compute $u(t,x)$ explicitly.

(iii) Explain in what sense the solution $u(t)$ from (ii) exists.

**Hint:** You may use $\big(\mathcal{F}\mathrm{e}^{-\frac{\lambda}{2}x^2}\big)(\xi) = \lambda^{-1/2}\,\mathrm{e}^{-\frac{\xi^2}{2\lambda}}$ and $\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{e}^{-\frac{\lambda}{2}x^2} = \sqrt{\frac{2\pi}{\lambda}}$ where $\lambda > 0$.

**Solution:**

(i) After applying the Fourier transform, we obtain

$$\mathcal{F}\big(\partial_t u(t)\big) = \partial_t \hat{u}(t) \overset{[1]}{=} \mathcal{F}\Big(\tfrac{1}{2}\partial_x^2 u(t) + f(t)\Big) \overset{[1]}{=} -\tfrac{1}{2}\xi^2\,\hat{u}(t) + \hat{f}(t)$$

where we have abbreviated the Fourier transforms of $u(t)$, $u_0$ and $f(t)$ with $\hat{u}(t)$, $\hat{u}_0$ and $\hat{f}(t)$. The solution to the *homogeneous* heat equation after Fourier transform is

$$\hat{u}(t) = \mathrm{e}^{-\frac{t}{2}\xi^2}\,\hat{u}_0. \qquad\qquad [1]$$

Hence, undoing the Fourier transform yields

$$u(t) \overset{[1]}{=} \mathcal{F}^{-1}\Big(\mathrm{e}^{-\frac{t}{2}\xi^2}\,\hat{u}_0\Big) \overset{[1]}{=} (2\pi)^{-1/2}\,\mathcal{F}^{-1}\big(\mathrm{e}^{-\frac{t}{2}\xi^2}\big) * u_0 \overset{[1]}{=} \frac{\mathrm{e}^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} * u_0 =: G(t) * u_0.$$

The solution to the *inhomogeneous* heat equation now is

$$u(t) \overset{[1]}{=} G(t) * u_0 + \int_0^t \mathrm{d}s\,G(t-s) * f(s).$$

(ii) For the special case $u_0 = 0$ and $f(t,x) = x$, we obtain

$$\begin{aligned}
u(t,x) &\overset{[1]}{=} 0 + \int_0^t \mathrm{d}s\,\big(G(t-s) * x\big)(x) \\
&\overset{[1]}{=} \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y\,(x-y)\,\frac{\mathrm{e}^{-\frac{y^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \\
&\overset{[1]}{=} \int_0^t \mathrm{d}s\,\left(x\int_{\mathbb{R}} \mathrm{d}y\,\frac{\mathrm{e}^{-\frac{y^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} - (t-s)\int_{\mathbb{R}} \mathrm{d}y\,\frac{y}{t-s}\,\frac{\mathrm{e}^{-\frac{y^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\right) \\
&\overset{[1]}{=} \int_0^t \mathrm{d}s\,x - \int_0^t \mathrm{d}s\,(t-s)\int_{\mathbb{R}} \mathrm{d}y\,\frac{\partial}{\partial y}\,\frac{\mathrm{e}^{-\frac{y^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} \\
&= t\,x - \int_0^t \mathrm{d}s\,(t-s)\left[\frac{\mathrm{e}^{-\frac{y^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\right]_{-\infty}^{+\infty} \overset{[1]}{=} t\,x.
\end{aligned}$$

(iii) Clearly, $\big|u(t,x)\big| = t\,|x|$ grows linearly for large $|x|$, and thus, it cannot be integrable for $t > 0$ [1]. But it is polynomially bounded, and thus, $u(t)$ is a weak (or distributional) solution to the heat equation [1].

3. **Tempered distributions (23 points)**

(i) Explain in what sense $f(x) = \big(|x-3|+2\big)^2$ defines a tempered distribution.

(ii) Compute the first two distributional derivatives of $f(x) = \big(|x-3|+2\big)^2$.

(iii) Compute the distributional Fourier transform of $g(x) = x^2\,\mathrm{e}^{-\frac{x^2}{2}}$.

(iv) Define the translation operator $\big(T_y\varphi\big)(x) := \varphi(x-y)$ for $y \in \mathbb{R}$. Extend $T_y$ to the tempered distributions in such a way that $T_y\delta = \delta_y$.

**Hint:** You may use $\big(\mathcal{F}\mathrm{e}^{-\frac{\lambda}{2}x^2}\big)(\xi) = \lambda^{-1/2}\,\mathrm{e}^{-\frac{\xi^2}{2\lambda}}$ and $\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{e}^{-\frac{\lambda}{2}x^2} = \sqrt{\frac{2\pi}{\lambda}}$ where $\lambda > 0$.

**Solution:**

(i) The distribution $f$ is defined via the integral

$$(f,\varphi) = \int_{\mathbb{R}} \mathrm{d}x\, f(x)\,\varphi(x) \qquad [1]$$

where $\varphi \in \mathcal{S}(\mathbb{R})$. Since $f$ is polynomially bounded, it defines a tempered distribution.

(ii) We note that $f(x) = (x-3)^2 + 4 + 4\,|x-3|$, and given that ordinary derivatives and weak derivatives of continuously differentiable functions coincide, we only need to compute the derivatives of $|x-3|$ [1]. The first derivative

$$\begin{aligned}
\big(\partial_x\,|x-3|\,,\varphi\big) &\overset{[1]}{=} -\big(|x-3|\,,\partial_x\varphi\big) \\
&\overset{[1]}{=} -\int_{\mathbb{R}} \mathrm{d}x\ |x-3|\ \partial_x\varphi(x) \\
&\overset{[1]}{=} +\int_{-\infty}^{3} \mathrm{d}x\,(x-3)\,\partial_x\varphi(x) - \int_{3}^{+\infty} \mathrm{d}x\,(x-3)\,\partial_x\varphi(x) \\
&\overset{[1]}{=} \Big[(x-3)\,\varphi(x)\Big]_{-\infty}^{3} - \int_{-\infty}^{3} \mathrm{d}x\,\varphi(x) - \Big[(x-3)\,\varphi(x)\Big]_{3}^{+\infty} + \int_{3}^{+\infty} \mathrm{d}x\,\varphi(x) \\
&\overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}x\,\mathrm{sgn}(x-3)\,\varphi(x) = \big(\mathrm{sgn}(x-3)\,,\varphi\big) \\
\big(\partial_x^2\,|x-3|\,,\varphi\big) &\overset{[1]}{=} -\big(\mathrm{sgn}(x-3)\,,\,\partial_x\varphi\big) \\
&\overset{[1]}{=} +\int_{-\infty}^{3} \mathrm{d}x\,\partial_x\varphi(x) - \int_{3}^{+\infty} \mathrm{d}x\,\partial_x\varphi(x) \\
&\overset{[1]}{=} \big[\varphi(x)\big]_{-\infty}^{3} - \big[\varphi(x)\big]_{3}^{+\infty} = 2\varphi(3) \\
&\overset{[1]}{=} \big(2\delta_3,\varphi\big)
\end{aligned}$$

So the two weak derivatives are:

$$\begin{aligned}
\partial_x f(x) &\overset{[1]}{=} 2(x-3) + 4\,\mathrm{sgn}(x-3) \\
\partial_x^2 f(x) &\overset{[1]}{=} 2 + 8\,\delta(x-3)
\end{aligned}$$

(iii) $g$ is a Schwartz function, and for integrable functions, distributional and conventional Fourier transform coincide [1]. Because the multiplication of the Gaußian $\mathrm{e}^{-\frac{x^2}{2}}$ by $x^2$ can be converted into taking a second derivative of the Fourier transform, we obtain

$$(\mathcal{F}g)(\xi) \overset{[1]}{=} \big(\mathcal{F}(x^2\,\mathrm{e}^{-\frac{x^2}{2}})\big)(\xi) \overset{[1]}{=} \mathrm{i}^2\,\partial_\xi^2\,\big(\mathcal{F}(\mathrm{e}^{-\frac{x^2}{2}})\big)(\xi)$$
$$\overset{[1]}{=} -\partial_\xi^2 e^{-\frac{\xi^2}{2}} = +\partial_\xi\big(\xi\,\mathrm{e}^{-\frac{\xi^2}{2}}\big)$$
$$\overset{[1]}{=} \big(1-\xi^2\big)\,\mathrm{e}^{-\frac{\xi^2}{2}}.$$

(iv) The correct extension is $\big(T_y L,\varphi\big) := \big(L, T_{-y}\varphi\big)$ [1], because then

$$\big(T_y\delta,\varphi\big) \overset{[1]}{=} \big(L, T_{-y}\varphi\big) \overset{[1]}{=} \int_{\mathbb{R}} \mathrm{d}x\,\delta(x)\,\varphi\big(x-(-y)\big)$$
$$\overset{[1]}{=} \varphi(y) \overset{[1]}{=} (\delta_y,\varphi).$$

4. **The free relativistic Schrödinger operator (22 points)**

Consider the multiplication operator $T$ defined through

$$(T\widehat{\psi})(\xi) := \sqrt{m^2+\xi^2}\,\widehat{\psi}(\xi).$$

(i) Show that $T$ is non-negative on $L^2(\mathbb{R}^d)$, i. e. $T \geq 0$.

(ii) Solve the free relativistic Schrödinger equation in momentum representation,

$$\mathrm{i}\,\partial_t\widehat{\psi}(t) = T\widehat{\psi}(t), \qquad \widehat{\psi}(0) = \widehat{\psi}_0 \in L^2(\mathbb{R}^d).$$

(iii) Define the relativistic kinetic energy operator in position representation

$$H := \mathcal{F}^{-1}\,T\,\mathcal{F}$$

in terms of the operator $T$ and the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$. Find the solution $\psi(t)$ to the Schrödinger equation in position representation,

$$\mathrm{i}\,\partial_t\psi(t) = H\psi(t), \qquad \psi(0) = \psi_0 \in L^2(\mathbb{R}^d).$$

(iv) Prove that the solution $\psi(t)$ of (iii) satisfies $\left\lVert\psi(t)\right\rVert_{L^2(\mathbb{R}^d)} = \lVert\psi_0\rVert_{L^2(\mathbb{R}^d)}$.

(v) Show that $H$ is symmetric on $\mathcal{S}(\mathbb{R}^d)$, i. e. $\langle\varphi, H\psi\rangle = \langle H\varphi, \psi\rangle$ holds for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$.

**Solution:**

(i) $T \geq 0$ means $\langle\widehat{\psi}, T\widehat{\psi}\rangle \geq 0$ holds for all $\widehat{\psi} \in L^2(\mathbb{R}^d)$ [1]:

$$\langle\widehat{\psi}, T\widehat{\psi}\rangle \overset{[1]}{=} \int_{\mathbb{R}^d}\mathrm{d}\xi\,\overline{\widehat{\psi}(\xi)}\,\bigl(T\widehat{\psi}\bigr)(\xi) \overset{[1]}{=} \int_{\mathbb{R}^d}\mathrm{d}\xi\,\underbrace{\sqrt{m^2+\xi^2}\,\bigl|\widehat{\psi}(\xi)\bigr|^2}_{\geq 0} \overset{[1]}{\geq} 0$$

(ii) Since $T$ is a multiplication operator, the solution is $\widehat{\psi}(t,\xi) = \mathrm{e}^{-\mathrm{i}t\sqrt{m^2+\xi^2}}\,\widehat{\psi}_0(\xi)$ [1].

(iii) We apply the Fourier transform to the Schrödinger equation in position representation and obtain

$$\mathcal{F}\bigl(\mathrm{i}\,\partial_t\psi(t)\bigr) \overset{[1]}{=} \mathrm{i}\,\partial_t\widehat{\psi}(t) = \mathcal{F}\bigl(H\psi(t)\bigr) \overset{[1]}{=} T\,\mathcal{F}\psi(t) \overset{[1]}{=} T\widehat{\psi}(t)$$

with initial condition $\widehat{\psi}(0) = \mathcal{F}\psi_0 = \widehat{\psi}_0$.

The solution of the Schrödinger equation in momentum and position representation are related by the Fourier transform,

$$\psi(t) \overset{[1]}{=} \mathcal{F}^{-1}\,\mathrm{e}^{-\mathrm{i}t\sqrt{m^2+\xi^2}}\,\mathcal{F}\psi_0 = (2\pi)^{-d/2}\,\mathcal{F}^{-1}\Bigl(\mathrm{e}^{-\mathrm{i}t\sqrt{m^2+\xi^2}}\Bigr) * \psi_0.$$

(iv) The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ is unitary, $\mathcal{F}^{-1} = \mathcal{F}^*$, and thus norm-preserving, $\left\lVert\mathcal{F}\psi_0\right\rVert_{L^2(\mathbb{R}^d)} = \lVert\psi_0\rVert_{L^2(\mathbb{R}^d)}$ [1]. Hence, we deduce

$$\begin{aligned}
\left\lVert\psi(t)\right\rVert^2_{L^2(\mathbb{R}^d)} &\overset{[1]}{=} \left\lVert\mathcal{F}^{-1}\,\mathrm{e}^{-\mathrm{i}t\sqrt{m^2+\xi^2}}\,\mathcal{F}\psi_0\right\rVert^2_{L^2(\mathbb{R}^d)} \overset{[1]}{=} \left\lVert\mathrm{e}^{-\mathrm{i}t\sqrt{m^2+\xi^2}}\,\mathcal{F}\psi_0\right\rVert^2_{L^2(\mathbb{R}^d)} \\
&\overset{[1]}{=} \int_{\mathbb{R}^d}\mathrm{d}\xi\,\left|\mathrm{e}^{-\mathrm{i}t\sqrt{m^2+\xi^2}}\,(\mathcal{F}\psi_0)(\xi)\right|^2 \overset{[1]}{=} \int_{\mathbb{R}^d}\mathrm{d}\xi\,\bigl|(\mathcal{F}\psi_0)(\xi)\bigr|^2 \\
&\overset{[1]}{=} \left\lVert\mathcal{F}\psi_0\right\rVert^2_{L^2(\mathbb{R}^d)} \overset{[1]}{=} \lVert\psi_0\rVert^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

(v) Let $\varphi, \psi$ be arbitrary Schwartz functions. The unitarity of the Fourier transform as well as the fact that $\mathcal{F}$ maps Schwartz functions onto Schwartz functions yields

$$\begin{aligned}\big\langle \varphi, H\psi \big\rangle &\overset{[1]}{=} \Big\langle \varphi, \mathcal{F}^{-1}\, T\, \mathcal{F}\psi \Big\rangle \overset{[1]}{=} \big\langle \mathcal{F}\varphi, T\,\mathcal{F}\psi \big\rangle \\ &\overset{[1]}{=} \int_{\mathbb{R}^d} \mathrm{d}\xi \, \overline{(\mathcal{F}\varphi)(\xi)} \, \sqrt{m^2 + \xi^2}\, (\mathcal{F}\psi)(\xi) = \int_{\mathbb{R}^d} \mathrm{d}\xi \, \overline{\sqrt{m^2 + \xi^2}\, (\mathcal{F}\varphi)(\xi)}\, (\mathcal{F}\psi)(\xi) \\ &\overset{[1]}{=} \big\langle T\,\mathcal{F}\varphi, \mathcal{F}\psi \big\rangle \overset{[1]}{=} \Big\langle \mathcal{F}^{-1}\, T\, \mathcal{F}\varphi, \psi \Big\rangle \\ &\overset{[1]}{=} \big\langle H\varphi, \psi \big\rangle.\end{aligned}$$