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Grade

Last name	First name

Student id #	Major

Signature

	I	II
1		
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I .... First correction

II .... Second correction

University of Toronto
Department of Mathematics

Test 2
Differential Equations of Mathematical Physics
(APM 351 Y)

Max Lein

30 January 2014, 09:10–10:50, Sidney Smith Hall, SS 1074

Remarks:
Please verify the completeness of the exam: **4** problems

Time allotted: **90** minutes

Allowed aids: **none**

Only to be filled out by the instructor/TA:

Left room from .... to ....

Handed in early at ....

Remarks:

1. **Partial differential equation on $\mathbb{T}^2$ (12 points)**
Consider the PDE

$$\partial_{x_1}^8 u - 2\partial_{x_2}^6 u + 3u = f$$

on $\mathbb{T}^2$ with $f \in \mathcal{C}^2(\mathbb{T}^2)$.

(i) Give the definition of the discrete Fourier transform $\mathcal{F} : L^1(\mathbb{T}^d) \longrightarrow \ell^\infty(\mathbb{Z}^d)$ and its inverse. (You may assume that the sum converges.)

(ii) Find the solution u.

(iii) Investigate the smoothness of the solution, i. e. find the largest integer k so that $u \in \mathcal{C}^k(\mathbb{T}^2)$.

2. **The heat equation (14 points)**

Consider the one-dimensional heat equation

$$\partial_t u(t) = \tfrac{1}{2}\partial_x^2 u(t) + f(t), \qquad u(0) = u_0 \in L^1(\mathbb{R}),$$

with inhomogeneity f.

(i) Find the solution $u(t)$.

(ii) For the case $f(t,x) = x$ and $u_0 = 0$, compute $u(t,x)$ explicitly.

(iii) Explain in what sense the solution $u(t)$ from (ii) exists.

Hint: You may use $\big(\mathcal{F}\mathrm{e}^{-\frac{\lambda}{2}x^2}\big)(\xi) = \lambda^{-1/2}\,\mathrm{e}^{-\frac{\xi^2}{2\lambda}}$ and $\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{e}^{-\frac{\lambda}{2}x^2} = \sqrt{\frac{2\pi}{\lambda}}$ where $\lambda > 0$.

3. **Tempered distributions (23 points)**

(i) Explain in what sense $f(x) = \big(|x-3|+2\big)^2$ defines a tempered distribution.

(ii) Compute the first two distributional derivatives of $f(x) = \big(|x-3|+2\big)^2$.

(iii) Compute the distributional Fourier transform of $g(x) = x^2\, \mathrm{e}^{-\frac{x^2}{2}}$.

(iv) Define the translation operator $\big(T_y\varphi\big)(x) := \varphi(x-y)$ for $y \in \mathbb{R}$. Extend T_y to the tempered distributions in such a way that $T_y\delta = \delta_y$.

Hint: You may use $\big(\mathcal{F}\mathrm{e}^{-\frac{\lambda}{2}x^2}\big)(\xi) = \lambda^{-1/2}\,\mathrm{e}^{-\frac{\xi^2}{2\lambda}}$ and $\int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\frac{\lambda}{2}x^2} = \sqrt{\frac{2\pi}{\lambda}}$ where $\lambda > 0$.

4. **The free relativistic Schrödinger operator (22 points)**

Consider the multiplication operator T defined through

$$(T\widehat{\psi})(\xi) := \sqrt{m^2 + \xi^2}\,\widehat{\psi}(\xi).$$

(i) Show that T is non-negative on $L^2(\mathbb{R}^d)$, i. e. $T \geq 0$.

(ii) Solve the free relativistic Schrödinger equation in momentum representation,

$$\mathrm{i}\,\partial_t \widehat{\psi}(t) = T\widehat{\psi}(t), \qquad \widehat{\psi}(0) = \widehat{\psi}_0 \in L^2(\mathbb{R}^d).$$

(iii) Define the relativistic kinetic energy operator in position representation

$$H := \mathcal{F}^{-1}\,T\,\mathcal{F}$$

in terms of the operator T and the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$. Find the solution $\psi(t)$ to the Schrödinger equation in position representation,

$$\mathrm{i}\,\partial_t \psi(t) = H\psi(t), \qquad \psi(0) = \psi_0 \in L^2(\mathbb{R}^d).$$

(iv) Prove that the solution $\psi(t)$ of (iii) satisfies $\big\lVert \psi(t) \big\rVert_{L^2(\mathbb{R}^d)} = \lVert \psi_0 \rVert_{L^2(\mathbb{R}^d)}$.

(v) Show that H is symmetric on $\mathcal{S}(\mathbb{R}^d)$, i. e. $\big\langle \varphi, H\psi \big\rangle = \big\langle H\varphi, \psi \big\rangle$ holds for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$.