....
Grade

	I	II
1		
2		
3		
4		
5		
6		
$\sum$		

I ....
First correction

II ....
Second correction

Last name

First name

Student id #

Major

Signature

University of Toronto
Department of Mathematics

Test 3
Differential Equations of Mathematical Physics (APM 351 Y)

Max Lein

20 March 2014, 09:10–10:50, Sidney Smith Hall, SS 1074

Remarks:
Please verify the completeness of the exam: **6** problems

Time allotted: **100** minutes

Allowed aids: **none**

Only to be filled out by the instructor/TA:

Left room from .... to ....

Handed in early at ....

Remarks:

1. **The framework of quantum mechanics (12 points)**
   Consider a quantum particle moving in $\mathbb{R}^d$.

   (i) Give an example of a Schrödinger operator. Explain the physical meaning of each of the terms.

   (ii) State the Schrödinger equation.

   (iii) Give the notion of observable, state and dynamical equation.

   (iv) Show that $H = H^*$ implies $\left\| \mathrm{e}^{-\mathrm{i}tH} \psi \right\|^2 = \|\psi\|^2$ for all $t \in \mathbb{R}$.

   (v) Explain the significance of (iv) for the Born rule.

2. **The Birman-Schwinger principle (6 points)**

Consider the Schrödinger operator $H = -\Delta_x + V$ on $\mathbb{R}^d$ where $V \leq 0$ is a non-positive potential which decays at ∞, $\lim\limits_{|x|\to\infty} V(x) = 0$.

(i) Give the Birman-Schwinger operator K_E and state the Birman-Schwinger principle.

(ii) Give a sufficient condition on K_E for the absence of eigenvalues of H at $-E < 0$.

3. **Green's functions for $-\partial_x^2 + E$ (14 points)**
Consider the linear operator $L_E := -\partial_x^2 + E$ for $E > 0$ on $\mathbb{R}$. Define the function

$$R_E(x) := \frac{\mathrm{e}^{-\sqrt{E}\,|x|}}{2\sqrt{E}}.$$

(i) Compute $\left(-\partial_x^2 + E\right) R_E(x)$ in the sense of tempered distributions.

(ii) Find the Green's function $G(x, y)$ to the operator L_E.

(iii) Given $\varphi \in L^2(\mathbb{R})$, solve $L_E \psi = \varphi$ for ψ.

4. **Symmetric operators (4 points)**
Show that $H = -\partial_x^2$ is symmetric on

$$\mathcal{D} := \left\{\psi \in \mathcal{C}^2([0,1]) \;\mid\; \varphi(0) = 0 = \varphi(1)\right\} \subset L^2([0,1]).$$

5. **The quantum energy functional (15 points)**

Define the average energy

$$\mathcal{E}(\varphi) = \int_{\mathbb{R}} \mathrm{d}x \, \Big(\big|\partial_x \varphi(x)\big|^2 + V(x) \, \big|\varphi(x)\big|^2 \Big)$$

associated to the quantum hamiltonian $H = -\partial_x^2 + V$ and $\varphi \in \mathcal{S}(\mathbb{R})$ for the potential

$$V(x) = \begin{cases} -x & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases} . \tag{1}$$

Moreover, define the family of scaled Gaußians $\varphi_\lambda(x) := \pi^{-1/4} \sqrt{\lambda} \, \mathrm{e}^{-\frac{\lambda^2}{2} x^2}$ for $\lambda > 0$.

(i) Determine the expected value of the energy $E(\lambda) := \mathcal{E}(\varphi_\lambda)$.

(ii) Express $E(\lambda)$ as a power series in λ.

(iii) Use the quadratic approximation of $E(\lambda) = e_0 + \lambda \, e_1 + \lambda^2 \, e_2 + \mathcal{O}(\lambda^3)$ to minimize $E(\lambda)$ for small λ. Compute the minimum of $E(\lambda)$ up to $\mathcal{O}(\lambda^3)$.

(iv) Does this hamiltonian have a bound state? Justify your answer.

Hint: You may use $\big(\mathcal{F} \mathrm{e}^{-\frac{\lambda^2}{2} x^2}\big)(\xi) = \lambda^{-1} \, \mathrm{e}^{-\frac{\xi^2}{2\lambda^2}}$ and $\int_{\mathbb{R}} \mathrm{d}x \, \mathrm{e}^{-\lambda^2 x^2} = \frac{\sqrt{\pi}}{\lambda}$ where $\lambda > 0$.

6. **The spectrum of an operator (5 points)**

Let T be a bounded operator on a Hilbert space $\mathcal{H}_1$ and $U : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ a unitary between two Hilbert spaces. Show $\sigma(T) = \sigma\big(U\,T\,U^{-1}\big)$.