Topological Classification of Electromagnetic Media

in collaboration with Giuseppe De Nittis

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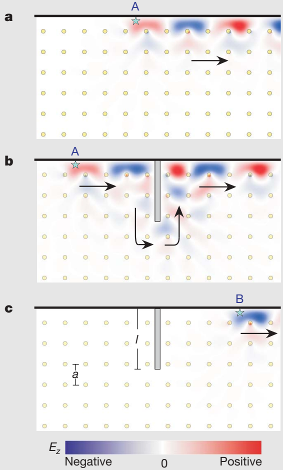
Quantum Hall Effect for Light

Predicted theoretically by Raghu & **Haldane** (2005) ...

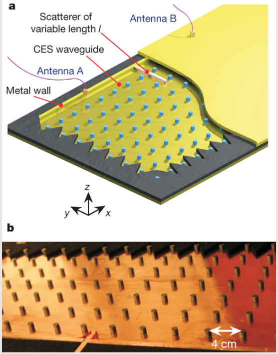
$$\left.\begin{matrix}\begin{pmatrix}\overline{\varepsilon} & 0 \\ 0 & \overline{\mu}\end{pmatrix} \neq \begin{pmatrix}\varepsilon & 0 \\ 0 & \mu\end{pmatrix} \\ \text{symmetry breaking}\end{matrix}\right\} \Longrightarrow$$



Joannopoulos, Soljačić et al (2009)

Quantum Hall Effect for Light

... and realized experimentally by Joannopoulos et al (2009)



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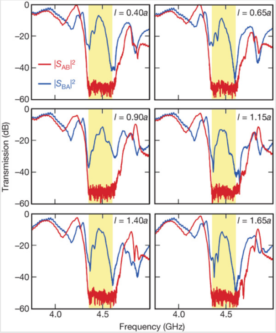

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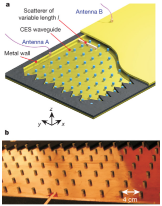
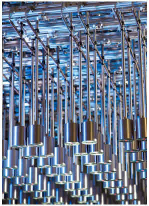
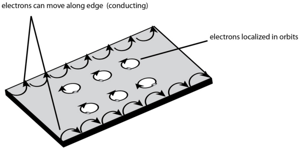
Topological Effects: Phenomenological Similarities



Light



Coupled Oscillators



Quantum

- Periodic structure ⇝ **bulk band gap**
- **Breaking** of time-reversal **symmetries**
- Unidirectional edge modes
- Robust under perturbations

Material vs. Crystallographic Symmetries

Material

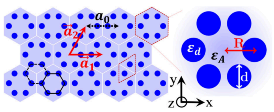
$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}$$

- Properties of and relations between ε, μ and χ
- Example:

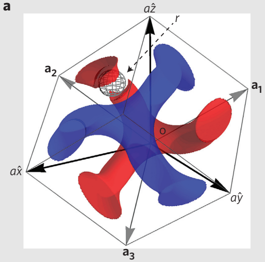
$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \neq \overline{W}, \quad \varepsilon \neq \mu$$

Only these are considered here!

Crystallographic



Wu & Hu (2015)



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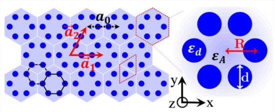

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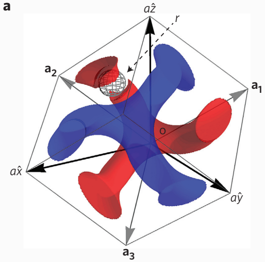


Lu et al (2013)

Two Main Questions for Today

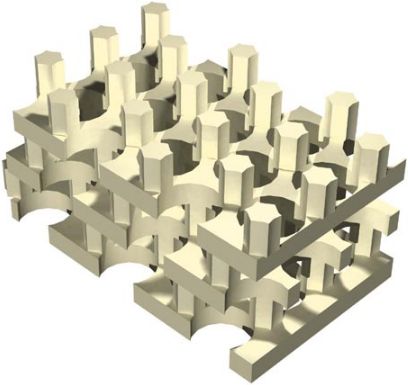
1. Is the **Quantum Hall Effect for Light really analogous** to the Quantum Hall Effect?
2. Are there **other topological effects**?

⇝ **Topological classification of electromagnetic media**

Strategy to Obtain Classification

1. Rewrite Maxwell's equations in the form of a Schrödinger equation
   (De Nittis & L., *The Schrödinger Formalism of Electromagnetism and Other Classical Waves* (2017))
2. Apply Cartan-Altland-Zirnbauer classification scheme of (quantum) topological insulators
   (De Nittis & L., *Symmetry Classification of Topological Photonic Crystals* (2017))

Setting: Electromagnetic Waves in Linear Medium



Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & \chi(x) \\ \chi(x)^* & \mu(x) \end{pmatrix}$$

1. dispersion-free
2. lossless
   ($W(x)^* = W(x)$ hermitian)
3. not a negative or 0 index material
   (the eigenvalues of $W(x)$ are positive and do not reach 0)
4. periodic
   ($W(x + \gamma) = W(x)$ for all lattice vectors γ)

Making Quantum-Wave Analogies Rigorous

Quantum Mechanics

$$\mathrm{i}\,\partial_t \Psi = H \Psi$$
$$H = (-\mathrm{i}\nabla - A)^2 + V$$

(Schrödinger equation)

$\longleftarrow ? \longrightarrow$

Classical Electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

(dynamical equations)

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

1. **States** describe the **configuration** of the system at a given time.
2. **Observables** represent experimentally **measurable** quantities.
3. **Dynamics** explain how states or observables **evolve over time**.

Recap: States and Dynamics in Quantum Mechanics

States and Dynamics

1. A **selfadjoint Hamilton operator**, e. g.

$$H = \frac{1}{2m}\big(-\mathrm{i}\nabla - A\big)^2 + V$$
$$H = m\,\beta + \big(-\mathrm{i}\nabla - A\big)\cdot\alpha + V$$

2. A **Hilbert space** $\mathcal{H}$ and states are represented by its elements, e. g. $L^2(\mathbb{R}^d, \mathbb{C}^n)$ with $\langle\phi,\psi\rangle = \int_{\mathbb{R}^d}\mathrm{d}x\,\phi(x)\cdot\psi(x)$.

3. **Dynamics** given by the Schrödinger equation

$$\mathrm{i}\,\partial_t\psi(t) = H\psi(t), \qquad \psi(0) = \phi$$

Schrödinger Formalism of Electromagnetism

States and Dynamics

1. **"Hamilton" operator** $M_+ = W^{-1}\,\mathrm{Rot}\big|_{\omega>0} = M_+^{*_W}$ where

$$\mathrm{Rot} = \begin{pmatrix} 0 & +\mathrm{i}\nabla^\times \\ -\mathrm{i}\nabla^\times & 0 \end{pmatrix}$$

2. **Hilbert space** $\mathcal{H}_+ = \Big\{ \Psi \in L^2(\mathbb{R}^3, \mathbb{C}^6) \;\big|\; \Psi \text{ is } \omega > 0 \text{ state} \Big\}$ with energy scalar product

$$\langle \Phi, \Psi \rangle_W = \int_{\mathbb{R}^3} \mathrm{d}x\, \Phi(x) \cdot W(x) \Psi(x)$$

3. **Dynamics** given by **Schrödinger equation**

$$\mathrm{i}\,\partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H}) \in \mathcal{H}_+$$

4. **Real-valuedness** of physical solutions:

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = 2\mathrm{Re}\, \Psi_+(t)$$

Representing Real Fields as Complex Waves

Generalize idea from in vacuo Maxwell equations

Real solutions = linear combinations of **complex** waves of $\pm\omega(\pm k)$

$$\cos(k \cdot x - \omega t) = \frac{1}{2}\Big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)} + \mathrm{e}^{-\mathrm{i}(k \cdot x - t\omega)}\Big) = \mathrm{Re}\,\big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)}\big)$$

$$\sin(k \cdot x - \omega t) = \frac{1}{\mathrm{i}2}\Big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)} - \mathrm{e}^{-\mathrm{i}(k \cdot x - t\omega)}\Big) = \mathrm{Im}\,\big(\mathrm{e}^{+\mathrm{i}(k \cdot x - t\omega)}\big)$$

Idea

Uniquely and systematically represent real, transversal fields as complex waves of $\omega > 0$

$$(\mathbf{E}, \mathbf{H}) = \Psi_+ + \Psi_- = 2\mathrm{Re}\,\Psi_+$$

Information contained in $\Psi_- = \overline{\Psi_+}$ is *redundant*

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Fundamental Equations

Maxwell's equations in media

1. *Maxwell's equations*

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}+\nabla\times\mathbf{H}\\ -\nabla\times\mathbf{E}\end{pmatrix} - \begin{pmatrix}J^D\\ J^B\end{pmatrix} \qquad \text{(dynamical eqns.)}$$

$$\begin{pmatrix}\nabla\cdot\mathbf{D}\\ \nabla\cdot\mathbf{B}\end{pmatrix} = \begin{pmatrix}\rho^D\\ \rho^B\end{pmatrix} \qquad \text{(constraint eqns.)}$$

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = \mathcal{W}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge*

$$\nabla\cdot J^\sharp + \partial_t\rho^\sharp = 0, \quad \sharp = D, B$$

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$$\begin{pmatrix}\nabla\cdot\mathbf{D}\\ \nabla\cdot\mathbf{B}\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \qquad \text{(constraint eqns.)}$$

2. *Constitutive relations*

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix} = W\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

3. *Conservation of charge* $\rightsquigarrow$ neglect sources for simplicity

$$\nabla\cdot J^{\sharp} + \partial_t\rho^{\sharp} = 0, \quad \sharp = D, B$$

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2. *Constitutive relations* for a linear, dispersive medium

$$\begin{pmatrix}\mathbf{D}(t,x)\\ \mathbf{B}(t,x)\end{pmatrix} = \int_{-\infty}^{t} \mathrm{d}s\, W(t-s,x) \begin{pmatrix}\mathbf{E}(s,x)\\ \mathbf{H}(s,x)\end{pmatrix}$$

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2. *Constitutive relations*

$$\big(\mathbf{D}(t),\mathbf{B}(t)\big) = \big(W * (\mathbf{E},\mathbf{H})\big)(t)$$

3. *Conservation of charge*

$$\nabla\cdot J^\sharp + \partial_t\rho^\sharp = 0, \quad \sharp = D, B$$

Heuristically Neglecting Dispersion in Maxwell's Equations

$$\mathrm{i}\,\frac{\partial}{\partial t} W * \Psi(t) = \mathrm{Rot}\,\Psi(t)$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$\omega\,\widehat{W}(\omega)\,\widehat{\Psi}(\omega) = \mathrm{Rot}\,\widehat{\Psi}(\omega)$$

$$\Big\downarrow \approx$$

$$\pm\,\omega\,\widehat{W}(\pm\omega_0)\,\widehat{\Psi}(\pm\omega) = \mathrm{Rot}\,\widehat{\Psi}(\pm\omega)$$

$$\Big\downarrow \mathcal{F}$$

$$\widehat{W}(\pm\omega_0)\,\mathrm{i}\,\frac{\partial}{\partial t}\Psi_\pm(t) = \mathrm{Rot}\,\Psi_\pm(t)$$

1. Apply **inverse Fourier transform** in time to go from time-dependent to frequency-dependent equations.
2. Approximate material weights $\widehat{W}(\pm\omega) \approx \widehat{W}(\pm\omega_0) = W_\pm$ for frequencies $\pm\omega \approx \pm\omega_0$.
   $+\omega_0$ *and* $-\omega_0$ contributions necessary to reconstruct **real** solutions.
3. **Undo Fourier transform** to obtain dynamical equations in the **absence of dispersion**.

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Dispersion-Free Maxwell Equations for Gyrotropic Media

Real solutions linear combination of complex $\pm\omega$ waves:

$$(\mathbf{E}, \mathbf{H}) = \Psi_+ + \Psi_- = 2\mathrm{Re}\,\Psi_\pm$$

$\Longrightarrow$ Pair of equations

$$\omega > 0 : \quad \begin{cases} W_+ \, \mathrm{i}\partial_t \Psi_+ = \mathrm{Rot}\,\Psi_+ \\ \mathrm{Div}\, W_+ \, \Psi_+ = 0 \end{cases}$$

$$\omega < 0 : \quad \begin{cases} W_- \, \mathrm{i}\partial_t \Psi_- = \mathrm{Rot}\,\Psi_- \\ \mathrm{Div}\, W_- \, \Psi_- = 0 \end{cases}$$

$$W = \overline{W} \quad \Longleftrightarrow \quad W_- = \overline{W_+}$$

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Schrödinger Formalism of Maxwell's Equations

Theorem (De Nittis & L. (2017))

$$\left.\begin{array}{c}\textbf{Real}\text{ transversal states} \\ (\mathbf{E},\mathbf{H}) = 2\mathrm{Re}\,\Psi_+ \\ \begin{pmatrix}\varepsilon & \chi \\ \chi^* & \mu\end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix}\psi_+^E \\ \psi_+^H\end{pmatrix} = \begin{pmatrix}+\nabla\times\psi_+^E \\ -\nabla\times\psi_+^H\end{pmatrix}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textbf{Complex}\text{ states with } \omega > 0 \\ \Psi = P_+(\mathbf{E},\mathbf{H}) \\ M = W^{-1}\,\mathrm{Rot}\,|_{\omega>0} = M^{*_W} \\ \mathrm{i}\,\partial_t\Psi = M\Psi\end{array}\right.$$

$$\mathcal{H} = \left\{\Psi \in L^2(\mathbb{R}^3,\mathbb{C}^6) \;\middle|\; \Psi \text{ is } \omega > 0 \text{ state}\right\}$$

$$\langle\Phi,\Psi\rangle_W = \int_{\mathbb{R}^3}\mathrm{d}x\,\Phi(x)\cdot W(x)\Psi(x)$$

Energy scalar product

(De Nittis & L., *The Schrödinger Formalism of Electromagnetism and Other Classical Waves* (2017))

Schrödinger Formalism of Electromagnetism

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2. **Hilbert space** $\mathcal{H}_+ \subset L^2_W(\mathbb{R}^3, \mathbb{C}^6)$
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$$\mathrm{i} \, \partial_t \Psi_+(t) = M_+ \Psi_+(t), \qquad \Psi_+(0) = P_+(\mathbf{E}, \mathbf{H}) \in \mathcal{H}_+$$

4. **Real-valuedness** of physical solutions:

$$\big(\mathbf{E}(t), \mathbf{H}(t)\big) = 2 \mathrm{Re} \, \Psi_+(t)$$

Note

This also applies to **gyrotropic** materials where $W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \neq \overline{W}$.

Application: Derivation of Ray Optics Equations

$$\begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \psi_+^E \\ \psi_+^H \end{pmatrix} = \begin{pmatrix} -\nabla \times \psi_+^E \\ +\nabla \times \psi_+^H \end{pmatrix}$$

(dynamical equation)

$$\begin{pmatrix} \nabla \cdot \\ \nabla \cdot \end{pmatrix} \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} \begin{pmatrix} \psi_+^E \\ \psi_+^H \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(constraint equation)

$$\xrightarrow{\lambda \ll 1} \begin{cases} \dot{r} = +\nabla_k \Omega + \mathcal{O}(\lambda) \\ \dot{k} = -\nabla_r \Omega + \mathcal{O}(\lambda) \end{cases}$$

(ray optics equations)

Setting

- Perturbation parameter $\lambda \ll 1$
- Slowly varying *electric permittivity* $\varepsilon = \varepsilon(\lambda)$ and *magnetic permeability* $\mu = \mu(\lambda)$ are 3×3-matrix-valued
- ε, μ and χ: periodic to "leading order"

Application: Derivation of Ray Optics Equations

$$\left.\begin{array}{c}\begin{pmatrix}\varepsilon & \chi \\ \chi^* & \mu\end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix}\psi_+^E \\ \psi_+^H\end{pmatrix} = \begin{pmatrix}-\nabla \times \psi_+^E \\ +\nabla \times \psi_+^H\end{pmatrix} \\ \text{(dynamical equation)} \\ \\ \begin{pmatrix}\nabla\cdot \\ \nabla\cdot\end{pmatrix} \begin{pmatrix}\varepsilon & \chi \\ \chi^* & \mu\end{pmatrix} \begin{pmatrix}\psi_+^E \\ \psi_+^H\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix} \\ \text{(constraint equation)}\end{array}\right\} \xrightarrow{\lambda \ll 1} \begin{cases}\dot{r} = +\nabla_k \Omega + \mathcal{O}(\lambda) \\ \dot{k} = -\nabla_r \Omega + \mathcal{O}(\lambda) \\ \text{(ray optics equations)}\end{cases}$$

Theorem (De Nittis & L. (2016))

- $\Omega(r, k) = \tau(r)\, \omega_n(k) + \mathcal{O}(\lambda)$ *and eom computed explicitly*
- *Depending on type of observables: Berry curvature enters*

(De Nittis & L., *Derivation of Ray Optics Equations in Photonic Crystals Via a Semiclassical Limit*, Ann. Henri Poincaré **18**, Issue 5, 2017)

Classification of Topological Insulators

1. **Topological class** $\longleftrightarrow$ Discrete symmetries of M
2. $\left.\begin{array}{c}\textbf{Phases}\text{ inside each}\\ \text{topological class}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Labeled by}\\ \text{topological invariants}\end{array}\right.$
3. **Bulk-edge correspondences**

$$\left.\begin{array}{c}\text{physical}\\ \text{observable}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{topological}\\ \text{invariant}\end{array}\right.$$

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No Additional Symmetries Assumption

Assumption

Apart from those below the system (i. e. the Maxwell operator M) has **no additional unitary, commuting symmetries**.

Otherwise

1. Block-decompose according to unitary, commuting symmetry.
2. Repeat until no extraneous symmetries are left.
3. Analyze each block separately with the tools used here.

Symmetries Used in Classification

Example

$$T_3 = (\sigma_3 \otimes \mathbb{1})\, C : \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \mapsto \begin{pmatrix} +\mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} C \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} +\overline{\mathbf{E}} \\ -\overline{\mathbf{H}} \end{pmatrix}$$

Pauli matrix $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ in electro-magnetic splitting

Symmetries Used in Classification

Unitary symmetries

$$U_n = \sigma_n \otimes \mathbb{1}, \quad n = 1, 2, 3$$

Antiunitary symmetries

$$T_n = (\sigma_n \otimes \mathbb{1})\, C, \quad n = 0, 1, 2, 3$$

- C is complex conjugation
- $\sigma_0 = \mathbb{1}$ the identity
- σ_1, σ_2 and σ_3 are the Pauli matrices in the **E**-**H** splitting
- U_n and T_n (anti)commute with free Maxwell operator

$$\mathrm{Rot} = \begin{pmatrix} 0 & +\mathrm{i}\nabla^{\times} \\ -\mathrm{i}\nabla^{\times} & 0 \end{pmatrix} = -\sigma_2 \otimes \nabla^{\times}$$

Admissible Symmetries

$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix} = \sum_{n=0}^{3} \sigma_n \otimes w_n$$

where e. g. $w_0 = \frac{1}{2}(\varepsilon + \mu)$ and $w_3 = \frac{1}{2}(\varepsilon - \mu)$

Admissible Symmetries

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Symmetry $V =$	$w_0 =$	$w_1 =$	$w_2 =$	$w_3 =$	*Symmetry Type*
$T_1 = (\sigma_1 \otimes \mathbb{1}) C$	Re w_0	Re w_1	Re w_2	i Im w_3	+TR
$U_2 = \sigma_2 \otimes \mathbb{1}$	w_0	0	w_2	0	ordinary
$T_3 = (\sigma_3 \otimes \mathbb{1}) C$	Re w_0	i Im w_1	Re w_2	Re w_3	+TR

Admissibility Conditions

- Reality of $(\mathbf{E}, \mathbf{H}) \iff \omega > 0$ fields $\mapsto \omega > 0$ fields $\implies V\,M = M\,V$
- Compatibility with energy scalar product $\implies V\,W = W\,V$

$\implies$ exclude *anti*commuting symmetries

Admissible Symmetries

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Admissibility Conditions

$\Longrightarrow$ exclude *anti*commuting symmetries

Relevance to Classification

$\Longrightarrow$ exclude **unitary, commuting** symmetries

Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Non-gyrotropic

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix}$$

$T_3 = (\sigma_3 \otimes \mathbb{1})\, C$

Gyrotropic

$$W = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \neq \begin{pmatrix} \overline{\varepsilon} & 0 \\ 0 & \overline{\mu} \end{pmatrix}$$

No symmetries

Dual-symmetric

$$W = \begin{pmatrix} \varepsilon & -\mathrm{i}\chi \\ +\mathrm{i}\chi & \varepsilon \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} & -\mathrm{i}\,\overline{\chi} \\ +\mathrm{i}\,\overline{\chi} & \overline{\varepsilon} \end{pmatrix}$$

$T_1 = (\sigma_1 \otimes \mathbb{1})\, C, \; T_3 = (\sigma_3 \otimes \mathbb{1})\, C$

“**EH**-symmetric”

$$W = \begin{pmatrix} \varepsilon & \chi \\ \chi & \varepsilon \end{pmatrix} = \begin{pmatrix} \overline{\varepsilon} & \overline{\chi} \\ \overline{\chi} & \overline{\varepsilon} \end{pmatrix}$$

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Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Non-gyrotropic

Class AI

Realized, e. g. dielectrics

Gyrotropic

Class A (Quantum Hall Class)

Realized, e. g. YIG for microwaves

Dual-symmetric

Two +TR $\Longrightarrow$ $2 \times$ Class AI

Realized, e. g. vacuum and YIG

[Khanikaev et al, Nature **12** (2013)]

"EH-symmetric"

Class AI

Realized?

(No example known to us.)

4 different topological classes of EM media

Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Non-gyrotropic	Gyrotropic
Class AI	Class A **(Quantum Hall Class)**
Realized, e. g. dielectrics	Realized, e. g. YIG for microwaves
Dual-symmetric	"**EH**-symmetric"
Two +TR $\Longrightarrow$ 2 $\times$ Class AI	Class AI
Realized, e. g. vacuum and YIG [Khanikaev et al, Nature **12** (2013)]	Realized? (No example known to us.)

Only one is topologically non-trivial in $d \leq 3$

Topological Classification of EM Media

Theorem (De Nittis & L. (2017))

Medium	*CAZ Class*	Dimension $d =$			
		1	2	3	4
Gyrotropic	A	0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6 \oplus \mathbb{Z}$
Non-gyrotropic	AI	0	0	0	$\mathbb{Z}$
EH-symmetric	AI	0	0	0	$\mathbb{Z}$
Dual-symmetric	$2 \times$ AI	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$

(Classification of Bloch vector bundles with symmetries.)

First and second Chern numbers

Topological Classification of EM Media

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First and second Chern numbers

Consequences of the Classification Result

1. Is the Quantum Hall Effect for Light really analogous to the Quantum Hall Effect?
   Answer: **Yes!** Both systems are in **Class A!**
2. Are there other topological effects?
   Answer: **In $d \leq 3$ (unfortunately) no!**
   (E. g. no analog of the Quantum Spin Hall Effect (class AII))
   In $d = 4$: Effects due to second Chern number or numbers?

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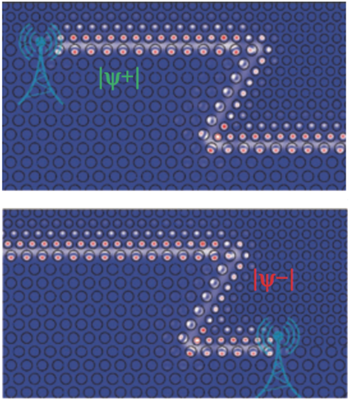
Comparison with Other Works

Unidirectional Modes of Fixed (Pseudo)spin

- Works of Xiao Hu et al and Aleksander Khanikaev et al
- Pseudospin degree of freedom in a **time-reversal-symmetric** medium
- "Hamiltonian" aka Maxwell operator $M = \begin{pmatrix} M_\uparrow & 0 \\ 0 & M_\downarrow \end{pmatrix}$ has a block decomposition
- Topological classification *must be applied to* $M_{\uparrow/\downarrow}$
- $M_{\uparrow/\downarrow}$ of (pseudo) spin ↑/↓ **may not possess** time-reversal symmetry

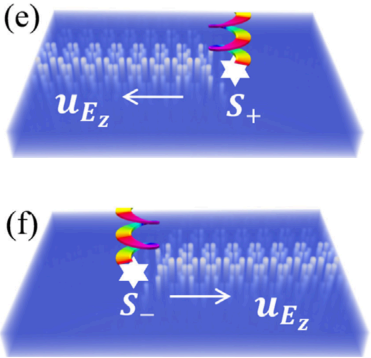


Khanikaev et al (2012)

Comparison with Other Works

Wu & Hu (2015)

- **Edge modes topological**
- Pseudospin degree of freedom in a **time-reversal-symmetric** medium
- Time-reversal symmetry $T_3 \neq T_\uparrow \oplus T_\downarrow$ **not blockdiagonal** $\Longrightarrow M_{\uparrow/\downarrow}$ class A (no symmetry)
- Chern numbers $C_\uparrow = -C_\downarrow \neq 0$ possible
- *Not* in contradiction, edge modes come in $\uparrow$ / $\downarrow$ **pairs**
- **Topologically protected** against perturbations which preserve T_3 symmetry and honeycomb structure

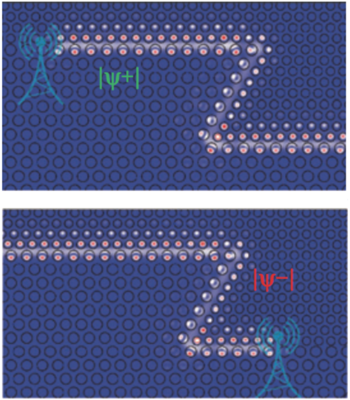


Wu & Hu (2015)

Comparison with Other Works

Khanikaev et al (2013)

- **Edge modes not topological**
- Dual-symmetric medium
- $T_3 = T_\uparrow \oplus T_\downarrow$ and $T_1 = (\mathrm{i}T_\uparrow) \oplus (-\mathrm{i}T_\downarrow)$ **blockdiagonal** and define equivalent symmetries on $\uparrow$ / $\downarrow$ subspaces
- Medium topologically trivial
- Boundary modes **for fixed spin come in pairs** (located at $\pm k$)



Khanikaev et al (2012)

Summary

- While Schrödinger formalism for non-gyrotropic media is well-known (1920s for vacuum, $\leq$1960s for non-gyrotropic media), for gyrotropic media it is new
- Schrödinger formalism = first order in time!
  $\Longrightarrow$ Systematic adaptation of quantum techniques to EM
- Restricting to $\omega > 0$ is conceptually essential, not just a technical footnote
- **Apart from gyrotropic media there are no other topologically non-trivial electromagnetic media**
- Ideas apply also to many other classical wave equations (e. g. certain acoustic waves)

Open Problems

- Give a mathematical proof of Haldane's photonic bulk-edge correspondence conjecture
  (dependence on boundary conditions, etc.)
- Non-linear effects
  (combining results by Babin & Figotin with De Nittis & L.)
- Application to other classical waves
- Crystallographic symmetries

Thank you for your attention!